NEPAL JOURNAL
OF
MATHEMATICAL SCIENCES
(NJMS)

Patron
Prof. Ram Prasad Khatiwada
Dean, Institute of Science and Technology, Tribhuvan University

Editor-in-Chief
Prof. Narayan Prasad Pahari
Director, School of Mathematical Sciences, Tribhuvan University, Kirtipur, Kathmandu, Nepal
Email: nppahari@gmail.com, njmseditor@gmail.com

Board of Editors
Narayan Adhikari, Central Department of Physics, Tribhuvan University, Nepal
Badri Adhikari, Department of Computer Science, University of Missouri-St. Louis, USA
Prakash Muni Bajracharya, School of Mathematical Sciences, Tribhuvan University, Nepal
Bal Krishna Bal, Department of Computer Engineering, Kathmandu University, Nepal
Debendra Banjade, Department of Mathematics, Coastal Carolina University, USA
Chet Raj Bhatta, Central Department of Mathematics, Tribhuvan University, Nepal
Ghanshyam Bhatt, Department of Mathematics, Tennessee State University, USA
Milan Bimali, Department of Biostatistics, University of Arkansas for Medical Sciences, USA
Mahananda Chalise, School of Management (SOM), Tribhuvan University
Ram Prasad Ghimire, School of Natural Sciences, Kathmandu University
Dil Bahadur Gurung, School of Natural Sciences, Kathmandu University
Dipak Kumar Jana, Applied Science, Haldia Institute of Technology, W.B., India
Durga Jang KC, Central Department of Mathematics, Tribhuvan University, Nepal
Vikash Kumar KC, Department of Statistics, Prithwi Narayan Campus, Tribhuvan University, Nepal
Shree Ram Khadka, Central Department of Mathematics, Tribhuvan University, Nepal
Shankar Prasad Khanal, Central Department of Statistics, Tribhuvan University, Nepal
Ganesh B. Malla, Department of Statistics, University of Cincinnati – Clermont, USA
Jyotsna Kumar Mandal, Department of Computer Sciences, Kalyani University, West Bangal, India
Danda Bir Rawat, Department of Computer Science, Howard University, Washington, DC, USA
Gyan Prakash Singh, Department of Statistics, Banaras Hindu University, India
Vikash Raj Satyal, Department of Statistics, Amrit Campus, Tribhuvan University, Nepal
Subarna Shakya, Institute of Engineering, Tribhuvan University, Pulchowk
Sahadeb Upretee, Department of Actuarial Science, Central Washington University, USA

Managing Coordinator
Mr. Keshab Raj Phulara, School of Mathematical Sciences, Tribhuvan University, Nepal
Acknowledgement

The Editorial Board would like to express their sincere gratitude and special thanks to the following reviewers for their substantial contribution of time and expertise to the journal’s rigorous editorial process for this volume, regardless of whether the papers are finally accepted /published or not.

- Sk. Golam Mortoja, Vidyasagar University, West Bengal, India
- Dipesh Barman, IIEST Shibpur, Kolkata, West Bengal, India
- Dil Bahadur Gurung, Kathmandu University, Nepal
- Amarendra Mishra, Patna University, Patna, India
- Shambhu Sharma, Dayalbagh Educational Institute, Agra, India
- Ram Prasad Khatriwada, Tribhuvan University, Nepal
- Durga Jang KC, Tribhuvan University, Nepal
- Parameshwari Kattel, Tribhuvan University, Nepal
- Khurram Shabbir, Department of Mathematics GC University, Lahore, Pakistan
- Gede Adhitya Wisnu Wardhana, Universitas Mataram, Indonesia
- Lakshmi Narayan Mishra, School of Advanced Sciences, Vellore Institute of Technology, Tamil Nadu, India.
- Lekhnath Bhattarai, Tribhuvan University, Nepal
- Prakash Upadhyay, Tribhuvan University, Nepal
- Shankar Prasad Khanal, Tribhuvan University, Nepal
- Eka Ratna Acharya, Tribhuvan University, Nepal
- Vinod Parajuli, Tribhuvan University, Nepal
- Debendra Banjade, Coastal Carolina University, USA

© School of Mathematical Sciences, Tribhuvan University

The views and interpretations in this journal are those of the author(s) and they are not attributable to the School of Mathematical Sciences, Tribhuvan University.

MAILING ADDRESS:
Nepal Journal of Mathematical Sciences
School of Mathematical Sciences,
Tribhuvan University, Kirtipur,
Kathmandu, Nepal
Email: nppahari@gmail.com, njmseditor@gmail.com
Editorial

Nepal Journal of Mathematical Sciences (NJMS) is the official peer-reviewed journal published by the School of Mathematical Sciences (SMS), Tribhuvan University. It is devoted to publishing the original research papers as well as critical survey articles in all parts of mathematical sciences and application at all levels including, but not limited to basic research leading to the development of new theories, techniques, and application to science, industry, and society. Its goal is to promote the exchange of ideas and information between all classes of mathematicians and/or the rest of society. This inaugural issue has been published both online and in print form. It includes nine research papers in different fields of mathematics and mathematical sciences. We acknowledge all the authors of this issue for publishing their scientific research works in this journal. We are highly grateful to all the eminent reviewers and editors for their enthusiastic involvement, cooperation, and support in this voluntary and painstaking work. Moreover, we would like to proceed with different scientific indexing shortly by promoting the quality of the journal. NJMS would like to increase the number of articles within the aims and scopes of the journal. Finally, we would like to request professors, scientists, and research scholars to contribute their original research works for the upcoming issues of the journal.

Editor-in-Chief

October 2020
Kathmandu, Nepal

Prof. Dr. Narayan Prasad Pahari
Nepal Journal of Mathematical Sciences (NJMS)
CONTENTS

1. A Novel Prey-Predator Quadratic Harvesting Model via Optimal Control Theory and Hopf Bifurcation
   Prabir Panja & Dipak Kumar Jana
   1-22

2. Existence and Extremal Solution of Boundary Value Problem for Nonlinear Hybrid Fractional Differential Equation in Banach Algebras
   B.D. Karande & Pravin M. More
   23-32

3. A New Two-Parameter Lindley Distribution
   Rama Shanker & Umme Habibah Rahman
   33-42

4. 3n+1 Problem and its Dynamics
   Bishnu Hari Subedi & Ajaya Singh
   43-50

5. An Alternative Proof of Rubin's Lemma
   Santosh Ghimire
   51-54

6. Determinants of Households' Adaptation Practices against Climate Change Impact on Off-farm Activities in Western Hill of Nepal
   Ananta Raj Dhungana, Vikash Kumar KC, Purna Bahadur Khand & Surya Mani Dhungana
   55-64

7. On The Degree of Approximation of a Function by Nörlund Means of its Fourier Laguerre Series
   Suresh Kumar Sahani, Vishnu Narayan Mishra & Narayan Prasad Pahari
   65-70

8. Basic Operations on Vedic Mathematics: A Study on Special Parts
   Krishna Kanta Parajuli
   71-76

9. A Note on Natural Transformation of Exton's Triple series Hypergeometric Function
   Harsh vardhan Harsh, Puneet Krishna Sharma & Shabana Khan
   77-84
A Novel Prey-Predator Quadratic Harvesting Model via Optimal Control Theory and Hopf Bifurcation

Prabir Panja & Dipak Kumar Jana
Department of Applied Science, Haldia Institute of Technology, Haldia
Purba Midnapur-721657, West Bengal, India
E-mail: prabirpanja@gmail.com, dipakjana@gmail.com
Corresponding Author: Prabir Panja

Abstract: In this investigation, a predator-prey interaction model among Phytoplankton, Zooplankton and Fish has been developed. In the absence of Zooplankton and Fish, it is assumed that Phytoplankton grows logistically. It is assumed that Zooplankton consumes Phytoplankton and Fish consumes Phytoplankton as well as Zooplankton. Holling type I & II functional responses have been considered to formulate the our proposed model. It is considered that Phytoplankton releases some toxin in the aquatic environment which makes some death in Zooplankton population. Quadratic harvesting is considered on Fish species. Boundedness of the solution of our proposed model has also been studied. Local stability of the system around each equilibrium point has been investigated. Also, the global stability of the interior equilibrium point has been studied. Existence condition of Hopf bifurcation of our proposed system has been studied. It is found that half saturation constant ($\alpha$) can change the system dynamics. It is also found that the harvesting rate of Fish (E) and consumption rate of Zooplankton ($\gamma_1$) has a significant role in the stability of the system. Again, it is found that the harvesting of Fish species will be increased if the selling price of Fish ($p$) and the annual discount ($\delta_1$) of Fish production cost increases. It is also found that the optimal harvesting rate of Fish decreases due to the increase of cost ($c$) of harvesting of Fish. Finally, some numerical simulation results have been presented to verify our analytical findings.

Keywords: Phytoplankton; Zooplankton; Fish; Optimal Control; Hopf bifurcation; Routh-Hurwitz criteria
1 Introduction

The study of prey-predator dynamics becoming the most important research topics for ecologists, scientists and applied mathematician due to its universal existence and importance. Mathematical modelling was started with the work of Volterra [32]. After Volterra, many researchers [5, 13, 17, 18, 19, 23, 24] have studied several mathematical models to understand predator-prey dynamics. Effects of seasonality on predator-prey dynamics has been investigated by Levy et al. [16]. Theoretical and evidence based predator-prey model has been studied by Abrams [2]. Wang and Jiang [33] investigated the chaos control of a delayed predator-prey model with the dormancy of predators. Bifurcation analysis of a time-delay model for prey-predator growth with stage-structure has been studied by Qu and Wei [26].

Mainly two types of plankton are found in the aquatic ecosystem such as Phytoplankton and Zooplankton. It is known that Phytoplankton are the primary producer in a food chain. Also, Phytoplankton makes food in the presence of sunlight with the help of chlorophyll. It is found that Phytoplankton forms bloom [6, 10, 20, 29] in the upper surface of the water. It is also shown that when Phytoplankton bloom take place the number of Phytoplankton sharply increases and decreases at short period of time and then returns its original very low level. Zooplankton are a type of heterotrophic plankton that ranges from microscopic organisms to large species, such as jellyfish. Zooplankton are found within large bodies of water, including oceans and freshwater systems. Zooplankton are drifting ecologically important organisms that are an integral component of the food chain. It is experimentally proved that some species of Phytoplankton produce some toxin [1, 6, 11, 12, 22, 28, 30] that makes some death of Zooplankton species. Zhang and Wang [34] have studied a nutrient-Phytoplankton-Zooplankton interaction predator prey model. They have shown that the system is persistent as long as the coexisting equilibrium exists. Shi and Yu [31] have developed a predator-prey model with one Phytoplankton species and two Zooplankton species including two types of delay. They have proved that the delays have a significant role in the stability of the Phytoplankton-Zooplankton system. Several research articles [7, 9, 15, 35] have been published on Phytoplankton-Zooplankton dynamics.

It is known that Fish provides a good source of high quality food that contains many vitamins and minerals. Fish is consumed by many animals as well as human beings throughout the world. It is observed that Fish consumes Phytoplankton as well as Zooplankton. Many people are depending on the harvesting on Fish. Clark [8] studied the optimal harvesting strategy of a particular species. So, the study of the dynamics of Fish in the presence of Phytoplankton and Zooplankton is very much important research topics. There exists very few research articles [4, 14, 21, 22, 27, 36] in the literature on Phytoplankton-Zooplankton-Fish
dynamics. But till now, several unexplored dynamics of Phytoplankton-Zooplankton-Fish to be investigated.

The rest of this paper is organized as follows: mathematical model is formulated in Section 2. Boundedness of all solutions of the proposed system is analyzed in Section 3. The existence of different equilibrium points has been determined in Section 4. Local stability analysis of our proposed system around each of the equilibrium points has been studied in Section 5. The global stability of interior equilibrium has been discussed in Section 6. Hopf bifurcation analysis has been done in Section 7. In Section 8, the Optimal control theory is applied to determine optimal harvesting rate of Fish. Numerical simulation results have been presented in section 9. Finally, in the last section we give the main outcomes of the present work.

2 Assumptions and Model Formulation

To study the dynamics of Phytoplankton, Zooplankton and Fish, the following assumptions have been made:

- It is considered that at time $t$, the densities of Phytoplankton, Zooplankton and Fish are $P(t)$, $Z(t)$ and $F(t)$ respectively.

- It is assumed that in the absence of Zooplankton and Fish, Phytoplankton growslogistically. Also, Zooplankton consumes Phytoplankton and Fish consumes Phytoplankton as well as Zooplankton.

- It is experimentally proved that consumption rate of Zooplankton by Fish is greater than the consumption rate of phytoplankton by Fish. For this reason, it is assumed that the consumption of Zooplankton and Phytoplankton by Fish have been followed Holling type I & II functional response respectively.

- It is assumed that Phytoplankton releases some toxin in the aquatic environment which makes some death of Zooplankton species.

- Optimal harvesting of Fish population has been considered. Also, Pontryagin’s maximum principle has been used to determine the optimal harvesting of Fish species.

- The intrinsic growth rate of Phytoplankton and the environmental carrying capacity of Phytoplankton have been considered as $r$ and $k$ respectively.
Again, the parameters $\beta$, $\beta_1$ and $d$ be taken as the consumption rate of Phytoplankton, the conservation rate of Phytoplankton and the natural death rate of Zooplankton respectively.

Also, it is assumed that Phytoplankton produces toxin which makes death of Zooplankton. The parameter $\rho$ denotes the rate of releasing the toxic substances produced by per unit biomass of Phytoplankton.

The parameters $\gamma$, $\gamma_1$, $s$ and $s_1$ denotes the consumption rate of Phytoplankton, the consumption rate of Zooplankton, conservation rate of Phytoplankton and the conservation rate of Zooplankton respectively.

Again, $\alpha$, $\delta$, $q$ and $E$ be the half saturation constant, the death rate of Fish, catchability coefficients and harvesting rate of Fish.

Keeping the above mentioned assumptions in mind, a three species interaction model of Phytoplankton, Zooplankton and Fish has been developed as follows:

$$
\begin{align*}
\frac{dP}{dt} &= rP(1 - \frac{P}{k}) - \frac{\beta PZ}{\alpha + P} - \frac{\gamma PF}{\alpha + P} \\
\frac{dZ}{dt} &= \frac{\beta_1 PZ}{\alpha + P} - dZ - \frac{\rho PZ}{\alpha + P} - \gamma ZF \\
\frac{dF}{dt} &= \frac{sPF}{\alpha + P} + s_1 ZF - \delta F - qEF^2
\end{align*}
$$

which satisfies the initial conditions $P(0) \geq 0$, $Z(0) \geq 0$ and $F(0) \geq 0$.

3 Boundedness of Solutions

In this section, Boundedness of all solutions of the system (1) has been investigated.

Theorem 1. Solutions of system (1) are bounded.

Proof. Let us define a function $W(t)$, as follows:

$$
W(t) = P(t) + Z(t) + F(t)
$$

Differentiating $W(t)$ with respect to time $t$, it is obtained that

$$
\frac{dW(t)}{dt} = \frac{dP(t)}{dt} + \frac{dZ(t)}{dt} + \frac{dF(t)}{dt}
$$

$$
= rP(1 - \frac{P}{k}) - \frac{\beta PZ}{\alpha + P} - \frac{\gamma PF}{\alpha + P} + \frac{\beta_1 PZ}{\alpha + P} - dZ - \frac{\rho PZ}{\alpha + P} - \gamma ZF \\
+ \frac{sPF}{\alpha + P} + s_1 ZF - \delta F - qEF^2
$$

$$
= rP(1 - \frac{P}{k}) - \frac{PZ}{\alpha + P}(\beta - \beta_1 + \rho) - \frac{PF}{\alpha + P}(\gamma - s) \\
- ZF(\gamma_1 - s_1) - \delta F - qEF^2
$$
Now we introduce a positive constant $\sigma$, then the above equation reduced as follows

$$
\frac{dW(t)}{dt} + \sigma W = rP\left(1 - \frac{P}{k}\right) - \frac{PZ}{\alpha + P}\left(\beta - \beta_1 + \rho\right) - \frac{PF}{\alpha + P}(\gamma - s)
- ZF(\gamma_1 - s_1) - \delta F - qEF^2 + \sigma P(t) + \sigma Z(t) + \sigma F(t)
\leq (r + \sigma)P - \frac{rP^2}{k} - (d - \sigma)Z - (\delta - \sigma)F - qEF^2
$$

Since $\beta > \beta_1 - \rho, \gamma > s, \gamma_1 > s_1$
\leq (r + \sigma)P - \frac{rP^2}{k} - qEF^2, \text{ Since } \sigma = \min\{d, \delta\}
\leq (r + \sigma)P - \frac{rP^2}{k} = f(P) \text{ (Say)}

Therefore,

$$
\max f(P) = \frac{k(r + \sigma)^2}{4r} = M(Say)
$$

i.e., \(\frac{dW}{dt} + \sigma W \leq \frac{k}{4r}(r + \sigma)^2\)

provided that $\sigma = \min(d, \delta)$.

Using the theory of differential inequality [3] in the above equation, it is obtained that

$$
0 < W(P, Z, F) < \frac{M}{\sigma}(1 - e^{-\sigma t}) + W(P(0), Z(0), F(0))e^{-\sigma t}
$$

where $M = \frac{k(r+\sigma)^2}{4r\sigma}$.

Now taking limit of the above inequality as $t$ tends to $\infty$, it is obtained that

$$
W(P, Z, F) \leq \frac{M}{\sigma}
$$

From the above equation, it is concluded that the solution of the system lies within the region

$$
\mathcal{G} = \{(P, Z, F)\in\mathbb{R}^3_+ : W \leq \frac{M}{\sigma} + \epsilon, \text{ for any } \epsilon > 0\}.
$$

Hence the proof.

\[\square\]

### 4 Equilibrium Points

In this section, different possible equilibrium points have been determined as follows:

- The trivial equilibrium point $E_0 = (0, 0, 0)$.
• The boundary equilibrium point \( E_1 = (k, 0, 0) \).

• The planer equilibrium point \( E_2 = (\hat{P}, \hat{Z}, 0) \) where
  \[
  \hat{P} = \frac{d\alpha}{\beta_1 - \rho - d} \quad \text{and} \quad \hat{Z} = \frac{r\alpha(\beta_1 - \rho)(k\beta_1 - k\rho - kd - \delta\alpha)}{k\beta_1 \beta_1 - \rho - kd - \delta\alpha \beta_1 - \rho}. 
  \]

• The another planer equilibrium point \( E_3 = (P', 0, F') \) where
  \[
  \begin{align*}
  P' &= \alpha(\delta + qEF') \div s - \delta - qE' \\
  F' &= \frac{r}{\alpha + \beta_1}(1 - P'k). 
  \end{align*}
  \]

• The positive equilibrium point \( E^* = (P^*, Z^*, F^*) \) where
  \[
  \begin{align*}
  F^* &= \frac{1}{\gamma} \left[ d - \frac{(\beta_1 - \rho)P^*}{\alpha + \beta_1} \right] \\
  Z^* &= \frac{1}{s_1} \left[ \delta + \frac{qEd}{\gamma} - \frac{P^*}{\alpha + \beta_1} \left( s + \frac{qE(\beta_1 - \rho)}{\gamma} \right) \right] \\
  P^* &= \text{satisfies the equation and } r \left( 1 - \frac{P^*}{k} \right) - \frac{\beta Z^*}{\alpha + \beta_1} - \frac{sF^*}{\alpha + \beta_1} = 0.
  \end{align*}
  \]

5 Local Stability Analysis

In this section, local stability of the proposed system (1) around the equilibrium points has been investigated. The stability of the equilibrium state is determined by the nature of the eigenvalues of the variational matrix \( J(P, Z, F) \) is given by

\[
J(P, Z, F) = \begin{pmatrix}
  r & -\beta P & -\gamma P \\
  \beta P & \frac{\alpha \beta Z}{\alpha + P^2} & \frac{\gamma P}{\alpha + P^2} \\
  \gamma P & \frac{\alpha \gamma F}{\alpha + P^2} & \frac{\alpha P}{\alpha + P^2}
\end{pmatrix}
\]

**Theorem 2.** The trivial equilibrium point \( E_0 \) is always unstable.

*Proof.* The variational matrix of the system (1) at \( E_0 \) is given by

\[
J_{E_0} = \begin{pmatrix}
  r & 0 & 0 \\
  0 & -d & 0 \\
  0 & 0 & -\delta
\end{pmatrix}
\]

It has eigenvalues \( \lambda_1 = r, \lambda_2 = -d \) and \( \lambda_3 = -\delta \). It is seen that one eigenvalue \( r > 0 \) is positive. So, we can say that the trivial equilibrium point is always unstable. Hence the proof.

**Theorem 3.** The boundary equilibrium point \( E_1 \) is locally asymptotically stable if \( k < \max \left\{ \frac{ad}{\beta_1 - \rho - d}, \frac{\delta a}{s - \delta} \right\} \).

*Proof.* The variational matrix of the system (1) at \( E_1 \) is given by

\[
J_{E_1} = \begin{pmatrix}
  -r & -\frac{\beta k}{\alpha + k} & -\frac{\gamma}{\alpha + k} \\
  0 & \frac{\beta k}{\alpha + k} - d - \frac{\rho k}{\alpha + k} & 0 \\
  0 & 0 & \frac{s k}{\alpha + k} - \delta
\end{pmatrix}
\]
The eigenvalues of the above variational matrix are \( \lambda_1 = -r, \lambda_2 = \frac{\beta_1 k}{\alpha + k} - d - \frac{\rho k}{\alpha + k} \) and \( \lambda_3 = \frac{sk}{\alpha + k} - \delta \). According to the condition of stability of a dynamical system, it is known to all that the equilibrium point \( E_1 \) is locally asymptotically stable whenever all the eigenvalues \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) must be less than zero. So for the stability of \( E_1, \lambda_2 < 0 \) and \( \lambda_3 < 0 \) must hold

\[
\begin{align*}
\text{i.e., } & \frac{\beta_1 k}{\alpha + k} - d - \frac{\rho k}{\alpha + k} < 0 \quad \text{and} \quad \frac{sk}{\alpha + k} - \delta < 0 \\
\text{i.e., } & \frac{k(\beta_1 - \rho - d) - d\alpha}{\alpha + k} < 0 \quad \text{and} \quad \frac{ks - \delta(k + \alpha)}{\alpha + k} < 0 \\
\text{i.e., } & k < \left\{ \frac{\alpha d}{\beta_1 - \rho - d} \right\} \quad \text{and} \quad k < \left\{ \frac{\delta \alpha}{s - \delta} \right\} \\
\text{i.e., } & k < \max \left\{ \frac{\alpha d}{\beta_1 - \rho - d}, \frac{\delta \alpha}{s - \delta} \right\}.
\end{align*}
\]

Hence the proof. \( \square \)

**Theorem 4.** The planer equilibrium point \( E_2 \) is locally asymptotically stable if \( B_1, B_3 > 0 \) and \( B_1B_2 > B_3 \) and otherwise it is unstable.

**Proof.** The variational matrix of system (1) at \( E_2 \) is given by

\[
J_{E_2} = \begin{pmatrix}
  b_1 & -b_2 & -b_3 \\
  b_4 & b_5 & -b_6 \\
  0 & 0 & b_7
\end{pmatrix}
\]

where \( b_1 = r - \frac{2x^2}{k} - \frac{\alpha \dot{Z}}{(\alpha + P)^2}, b_2 = \frac{\beta P}{\alpha + P}, b_3 = \frac{\alpha \dot{Z}}{(\alpha + P)^2} - \frac{\rho \dot{Z}}{\alpha + P}, b_4 = \frac{\beta_1 \dot{Z}}{(\alpha + P)^2} - \frac{\rho \dot{Z}}{\alpha + P}, b_5 = \frac{\beta_1 P}{\alpha + P} - d - \frac{\rho P}{\alpha + P}, b_6 = \gamma_1 \dot{Z} \) and \( b_7 = \frac{s P}{\alpha + P} + s_1 \dot{Z} - \delta - 2q E F \).

Then the characteristic equation of the above matrix is given by

\[
x^3 + B_1 x^2 + B_2 x + B_3 = 0
\]

where \( B_1 = -(b_1 + b_5 + b_7), B_2 = b_1 b_7 + b_3 b_7 + b_1 b_5 + b_3 b_4, B_3 = -b_1 b_5 b_7 - b_2 b_4. \) Now by Routh-Hurwitz criteria, the equilibrium point \( E_2 \) is locally asymptotically stable i.e., the eigenvalues of the characteristic equation may have negative real parts if \( B_1, B_3 > 0 \) and \( B_1B_2 - B_3 > 0 \) and unstable otherwise. \( \square \)

**Theorem 5.** Another planer equilibrium point \( E_3 \) is locally asymptotically stable if \( B'_1, B'_3 > 0 \) and \( B'_1B'_2 - B'_3 > 0 \) and unstable otherwise.

**Proof.** The variational matrix of the system (1) at \( E_3 \) is given by

\[
J_{E_3} = \begin{pmatrix}
  b_{11} & -b_{12} & -b_{13} \\
  0 & b_{22} & 0 \\
  b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]
where \( b_{11} = r - \frac{2rP'}{k} - \frac{\gamma_1 F'}{(a+P')^2} \), \( b_{12} = \frac{\beta P'}{(a+P')^2} \), \( b_{13} = \frac{\gamma P'}{(a+P')^2} \), \( b_{22} = \frac{\beta P'}{(a+P')^2} - d - \frac{\rho P'}{(a+P')^2} - \gamma_1 F' \), \( b_{31} = \frac{s_1 F'}{(a+P')^2} \), \( b_{32} = s_1 F' \) and \( b_{33} = \frac{s_1 F'}{(a+P')^2} - \delta - 2qEF' \).

The characteristic equation of the above matrix is given by

\[
y^3 + B_1'y^2 + B_2'y + B_3' = 0
\]

where \( B_1' = -(b_{11} + b_{22} + b_{33}), B_2' = b_{11}b_{33} + b_{22}b_{33} + b_{11}b_{22} + b_{13}b_{31} \) and \( B_3' = -b_{11}b_{22}b_{33} - b_{13}b_{31}b_{22} \). According to Routh-Hurwitz criteria, the system will be locally asymptotically stable i.e., the eigenvalues may be negative real parts if \( B_1', B_3' > 0 \) and \( B_1'B_3' - B_3'^2 > 0 \). Otherwise the system becomes unstable.

**Theorem 6.** The interior equilibrium point \( E^* \) is locally asymptotically stable if \( \sigma_1, \sigma_3 > 0 \) and \( \sigma_1\sigma_2 > \sigma_3 \) and otherwise it is unstable.

**Proof.** The variational matrix of system (1) at \( E^* \) is given by

\[
J_{E^*} = \begin{pmatrix}
\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\
\sigma_{21} & \sigma_{22} & -\sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\]

where \( \sigma_{11} = r - \frac{2rP^*}{K} - \frac{\alpha_1 Z^*}{(a+P^*)^2} - \frac{\gamma_1 F^*}{(a+P^*)^2}, \sigma_{12} = \frac{\beta P^*}{(a+P^*)^2}, \sigma_{13} = \frac{\gamma P^*}{(a+P^*)^2}, \sigma_{21} = \frac{\beta_1 Z^*}{(a+P^*)^2} - \frac{\beta_1 Z^*}{(a+P^*)^2}, \sigma_{22} = \frac{\beta_1 Z^*}{(a+P^*)^2} - \frac{\beta_1 Z^*}{(a+P^*)^2}, \sigma_{23} = \frac{\gamma_1 Z^*}{a+P^*}, \sigma_{31} = \frac{s_1 F^*}{(a+P^*)^2} - \frac{s_1 F^*}{(a+P^*)^2}, \sigma_{32} = s_1 F^* \) and \( \sigma_{33} = sP^* + s_1 Z^* - \delta - 2qEF^* \).

The characteristic equation of the above variational matrix \( J_{E^*} \) is given by

\[
x^3 + \sigma_1 x^2 + \sigma_2 x + \sigma_3 = 0.
\]  

where \( \sigma_1 = -(\sigma_{11} + \sigma_{22} + \sigma_{33}), \sigma_2 = \sigma_{22}\sigma_{33} + \sigma_{23}\sigma_{32} + \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{12}\sigma_{21} + \sigma_{13}\sigma_{31} \) and \( \sigma_3 = -\sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}\sigma_{32} - \sigma_{12}\sigma_{31}\sigma_{32} - \sigma_{12}\sigma_{31}\sigma_{32} - \sigma_{31}\sigma_{22}\sigma_{13} + \sigma_{13}\sigma_{21}\sigma_{32} \). According to Routh-Hurwitz criteria, the interior equilibrium is locally asymptotically stable if \( \sigma_1, \sigma_3 > 0 \) and \( \sigma_1\sigma_2 - \sigma_3 > 0 \) holds. Otherwise \( E^* \) is unstable.

## 6 Global Stability Analysis

In this section, the global stability analysis of the proposed system (1) around the equilibrium points \( E^*(P^*, Z^*, F^*) \) has been investigated.

**Theorem 7.** The system (1) is globally asymptotically stable around the interior equilibrium point \( E^*(P^*, Z^*, F^*) \) if \( \frac{\xi}{k} > \beta Z^* + \gamma F^* \) hold.
Proof. Let us choose a suitable Lyapunov function

\[ V = P - P^* \log \left( \frac{P}{P^*} \right) + k_1 \left( Z - Z^* \log \frac{Z}{Z^*} \right) + k_2 \left( F - F^* \log \frac{F}{F^*} \right) \]

Taking time derivative of the above equation, we have

\[ \frac{dV}{dt} = \frac{dP}{dt} \left( P - P^* \right) + k_1 \left( Z - Z^* \right) \frac{dZ}{dt} + k_2 \left( F - F^* \right) \frac{dF}{dt} \]

Substituting the value of \( \frac{dP}{dt}, \frac{dZ}{dt} \) and \( \frac{dF}{dt} \) in the above equation, it is obtained that

\[
\frac{dV}{dt} = (P - P^*) \left\{ r \left( 1 - \frac{P}{k^*} \right) - \frac{\beta Z}{\alpha + P} - \frac{\gamma F}{\alpha + P} \right\} + k_1 (Z - Z^*) \left\{ \frac{\beta_1 P}{\alpha + P} - d - \frac{\rho P}{\alpha + P} - \gamma_1 F \right\} \\
+ k_2 (F - F^*) \left\{ \frac{s P}{\alpha + P} + s_1 Z - \delta - q E F \right\}
\]

After simplification, it is obtained that

\[
\frac{dV}{dt} \leq \left( - \frac{r}{k^*} + \beta Z^* + \gamma F^* \right) (P - P^*)^2 - k_2 q E (F - F^*)^2 \\
+ (P - P^*) (Z - Z^*) \left\{ k_1 \alpha (\beta_1 - \rho) - \beta P^* - \beta \right\} \\
+ (P - P^*) (F - F^*) \left\{ k_2 s_1 - k_1 \right\}
\]

If we choose \( k_1 = \frac{\beta (\alpha + P^*)}{\alpha (\beta_1 - \rho)} \), \( k_2 = \frac{\gamma (\alpha + P^*)}{s \alpha} \) and \( k_2 s_1 = k_1 \gamma_1 \), then the above equation reduces the following form

\[
\frac{dV}{dt} \leq - \left( \frac{r}{k^*} - \beta Z^* - \gamma F^* \right) (P - P^*)^2 - k_2 q E (F - F^*)^2 \\
< 0, \text{ whenever } \frac{r}{k^*} > \beta Z^* + \gamma F^*
\]

Hence the proof. \( \square \)

7 Bifurcation Analysis

In this section, Hopf bifurcation analysis of system (1) around the interior equilibrium point has been discussed. The main objective of this investigation is to study the impacts of
change of model parameters on the dynamics of our proposed system. In this paper, we
have considered $\alpha$ as a bifurcation parameter and $\alpha^*$ represents the critical value or the
bifurcating value of the concerned bifurcation parameter.

**Theorem 8.** The positive equilibrium $E^*$ enters into Hopf bifurcation as $\alpha$ varies over $\mathbb{R}$.
Let $\phi : (0, \infty) \to \mathbb{R}$ be the following continuously differential function of $\alpha$.

$$
\phi(\alpha) = \sigma_1(\alpha)\sigma_2(\alpha) - \sigma_3(\alpha).
$$

Let $\alpha^*$ be a positive root of the equation $\phi(\alpha) = 0$. Therefore, the Hopf bifurcation of the
interior equilibrium $E(P^*, Z^*, F^*)$ occurs at $\alpha = \alpha^*$ if and only if

(i) $\phi(\alpha^*) = 0$

(ii) $L_2(\alpha^*)L_4(\alpha^*) + L_1(\alpha^*)L_3(\alpha^*) \neq 0$.

**Proof.** By the condition $\phi(\alpha) = 0$, then the characteristic equation of the variational matrix $V_{E^*}$ from **Theorem 6.** can be written as

$$(x^2 + \sigma_2)(x + \sigma_1) = 0$$

Let us considered the roots of the above equation are $\rho_1, \rho_2$ and $\rho_3$. Let the pair of imaginary roots at $\alpha = \alpha^*$ are $\rho_1, \rho_2$, then we have $\rho_3 = -\sigma_1$ and $\rho_1, \rho_2 = \pm i\sqrt{\sigma_2}$.

As $\psi(\alpha^*)$ is a continuous function of all its roots so there exists an open interval $(\alpha^* - \epsilon, \alpha^* + \epsilon)$ where $\rho_1$ and $\rho_2$ are complex conjugate for $\alpha$. Suppose that their general forms in this neighborhood are

$$
\rho_1(\alpha) = \chi(\alpha) + i\xi(\alpha)
$$

$$
\rho_2(\alpha) = \chi(\alpha) - i\xi(\alpha)
$$

Now we shall verify the transversality condition $\frac{d(Re\rho_1)}{d\alpha}|_{\alpha=\alpha^*} \neq 0, j = 1, 2$. Substituting, $\rho_j(\alpha) = \chi(\alpha) \pm i\xi(\alpha)$ into the characteristic equation and calculating the derivative, it is obtained that

$$
L_1(\alpha)\chi'(\alpha) - L_2(\alpha)\xi'(\alpha) + L_3(\alpha) = 0
$$

$$
L_2(\alpha)\chi'(\alpha) + L_1(\alpha)\xi'(\alpha) + L_4(\alpha) = 0
$$

where

$$
L_1(\alpha) = 3\chi^2 - 3\xi^2 + 2\sigma_1\chi + \sigma_2
$$

$$
L_2(\alpha) = 6\chi\xi + 2\sigma_1\xi
$$

$$
L_3(\alpha) = \sigma_1'\chi^2 - \sigma_2'\xi^2 + \sigma_2'\chi + \sigma_3'
$$

$$
L_4(\alpha) = 2\sigma_1'\chi\xi + \sigma_2'\xi
$$
Solving for $\xi'(\alpha)$ at $\alpha = \alpha^*$, it is obtained that

$$\frac{d(\text{Re} p_j(\alpha))}{d\alpha}|_{\alpha = \alpha^*} = \chi'(\alpha^*) = -\frac{L_2(\alpha^*)L_4(\alpha^*) + L_1(\alpha^*)L_3(\alpha^*)}{L_1^2(\alpha^*) + L_2^2(\alpha^*)} \neq 0$$

if $L_2(\alpha^*)L_4(\alpha^*) + L_1(\alpha^*)L_3(\alpha^*) \neq 0$. Thus the transversality condition holds and hence Hopf bifurcation occurs at $\alpha = \alpha^*$.

Hence the proof.

8 Optimal Harvesting

In this section, the optimal harvesting of Fish species has been determined. It is known to all that the harvesting is an important issue for the economic development of a country. Again, continuous and unscientific harvesting of a Fish species can extinct the species. So, the optimal harvesting is necessary for the conservation of Fish species. The emphasis of this section is on the profit making aspect of Fisheries. It is a thorough study of the optimal harvesting policy and the profit earned by harvesting considering quadratic costs and conservation of Fish species. The prime reason for using quadratic costs is that it allows to derive an analytic expression for the optimal harvesting. It is assumed that the price is a function which decreases with the increasing biomass. Thus to maximize the total discounted net revenues from the Fishery, the optimal control problem can be formulated as follows:

$$J(E) = \int_{t_0}^{t_f} e^{-\delta_1 t} \left[ (p - \omega q_1 E F^2)q_1 E F^2 - cE \right] dt$$

where $\delta_1$ is the annual discount rate of cost of Fish production, $p$ is the constant price per unit biomass of Fish, $c$ is the constant cost of the cost of harvesting of Fish species and $\omega$ is the economic constant. The problem (4) can be solved by applying Pontryagin’s maximum principle [25] subject to the model system (1) and control constraints $0 \leq E \leq E_{\text{max}}$.

The convexity of the objective function with respect to $E$, the linearity of the differential equations in the control and the compactness of the range values of the state variables can be combined to give the existence of the optimal control. Suppose $E_{\delta_1}$ is an optimal control with corresponding states $P_{\delta_1}, Z_{\delta_1}$ and $F_{\delta_1}$. We take $A_{\delta_1}(P_{\delta_1}, Z_{\delta_1}, F_{\delta_1})$ as optimal equilibrium point. Here, we intend to derive optimal control $E_{\delta_1}$ such that

$$J(E_{\delta_1}) = \max \{ J(E) : E \in U \}$$

where $U$ is the control set defined by $U = \{ E : [t_0, t_f] \to [0, E_{\text{max}}] | E \text{ is Lebesgue measurable} \}$.

The Hamiltonian of this optimal control problem is given by

$$H = \left\{ (p - \omega q_1 E F^2)q_1 E F^2 - cE \right\} + \lambda_1 \left\{ rP \left( 1 - \frac{P}{k} \right) - \frac{\beta P Z}{\alpha + P} - \frac{\gamma P F}{\alpha + P} \right\} + \lambda_2 \left\{ \frac{\beta_1 P Z}{\alpha + P} - dZ - \frac{c P Z}{\alpha + P} - \gamma_1 Z F \right\} + \lambda_3 \left\{ \frac{s P F}{\alpha + P} + s_1 Z F - \delta F - q E F^2 \right\}$$
where \( \lambda_i, i = 1, 2, 3 \) are the adjoint variables. The transversality condition for problem gives
\( \lambda_i(t_f) = 0, i = 1, 2, 3 \).
Now, it is possible to find the characterization of the optimal control \( E_{\delta_1} \). On the set
\( t : 0 < E_{\delta_1}(t) < E_{\text{max}} \), we have
\[
\frac{\partial H}{\partial E} = pq_1 F - 2 \omega q_1^2 F^4 E - c - q F^2 \lambda_3
\]
Thus at \( A_{\delta_1} (P_{\delta_1}, Z_{\delta_1}, F_{\delta_1}) \), \( E = E_{\delta_1}(t) \) and
\[
\frac{\partial H}{\partial E} = pq_1 F - 2 \omega q_1^2 F^4 E - c - q F^2 \lambda_3 = 0.
\]
This implies that
\[
E_{\delta_1} = \left\{ \frac{pq_1 F_{\delta_1} - c - q F^2_{\delta_1} \lambda_3}{2 \omega q_1^2 F^4_{\delta_1}} \right\}
\]
Now the adjoint equations at the point \( A_{\delta_1} (P_{\delta_1}, Z_{\delta_1}, F_{\delta_1}) \) are
\[
\frac{d\lambda_1}{dt} = \delta_1 \lambda_1 - \left( \frac{\partial H}{\partial P} \right)_{A_{\delta_1}}
= \delta_1 \lambda_1 - \lambda_1 \left\{ r \left( 1 - \frac{2P_{\delta_1}}{k} \right) - \frac{\beta Z_{\delta_1}}{(\alpha + P_{\delta_1})^2} - \frac{\gamma F_{\delta_1}}{(\alpha + P_{\delta_1})^2} \right\}
+ \lambda_2 \left\{ \frac{\beta_1 Z_{\delta_1}}{(\alpha + P_{\delta_1})^2} - \frac{\rho Z_{\delta_1}}{(\alpha + P_{\delta_1})^2} \right\} + \lambda_3 \left\{ \frac{s F_{\delta_1}}{(\alpha + P_{\delta_1})^2} \right\}
\]
\[
\frac{d\lambda_2}{dt} = \delta_1 \lambda_2 - \left( \frac{\partial H}{\partial Z} \right)_{A_{\delta_1}}
= \delta_1 \lambda_2 + \lambda_1 \left\{ \frac{\beta P_{\delta_1}}{\alpha + P_{\delta_1}} \right\} - \lambda_2 \left\{ \frac{\beta_1 P_{\delta_1}}{\alpha + P_{\delta_1}} - d - \frac{\rho P_{\delta_1}}{\alpha + P_{\delta_1}} - \gamma_1 F_{\delta_1} \right\} - \lambda_3 \left\{ s_1 F_{\delta_1} \right\}
\]
\[
\frac{d\lambda_3}{dt} = \delta_1 \lambda_3 - \left( \frac{\partial H}{\partial F} \right)_{A_{\delta_1}}
= \delta_1 \lambda_3 - \left\{ 2pq_1 F_{\delta_1} - 4Wq_1^2 E^2 F^4_{\delta_1} \right\} + \frac{\lambda_1 \gamma P_{\delta_1}}{\alpha + P_{\delta_1}} + \lambda_2 \gamma_1 Z_{\delta_1}
\]
\[
- \lambda_3 \left\{ \frac{s_1 P_{\delta_1}}{\alpha + P_{\delta_1}} - s_1 Z_{\delta_1} - \delta - 2q_1 E F_{\delta_1} \right\}
\]
The equations (5)-(7) are the first order system of simultaneous differential equations and it is easy to get the analytical solution of the equation with the help of initial conditions
\( \lambda_i(t_f) = 0, i = 1, 2, 3 \).

9 Numerical Illustrations

In this section, the dynamical behavior of the proposed model (1) has been investigated numerically using MATLAB software. Due to unavailability of real data, a set of hypothetical
values of parameters has been considered as $r = 2.0, k = 120, \beta = 0.6, \alpha = 4.0, \gamma = 0.5, \beta_1 = 0.4, d = 0.03654, \rho = 0.1, s = 0.2, s_1 = 0.2, \delta = 0.6, q = 0.01, E = 0.4, \gamma_1 = 0.1$. Using this set of parametric values, Figure 1 has been drawn. From this figure, it is seen that the interior equilibrium point is locally asymptotically stable. So, we can say that Phytoplankton, Zooplankton and Fish species coexist.

Using this set of parametric values $r = 2.0, k = 120, \beta = 0.6, \gamma = 0.5, \beta_1 = 0.9, d = 0.3654, \rho = 0.2, \gamma_1 = 0.6, s = 0.2, s_1 = 0.2, \delta = 0.6, q = 0.01, E = 0.1$, bifurcation diagram of system (1) with respect to $\alpha$ has been drawn in Figure 2. We have numerically calculated the critical value of $\alpha$ as 1.18. From this Figure, it is observed that the system loses it’s stability whenever $\alpha < 1.18$, but the system continues stable steady state behaviour for $\alpha > 1.18$. So, it can be concluded that the half saturation constant has a significant role on the stability of our proposed system.

![Figure 1: Time evolution for population densities.](image-url)
Bifurcation diagram of system (1) with respect to $E$ has been presented in Figure 3 for the set of parametric values $r = 0.4, k = 90, \beta = 0.6, \alpha = 1.0, \gamma = 0.5, \beta_1 = 0.9, d = 0.3654, \rho = 0.2, \gamma_1 = 0.6, s = 0.4, s_1 = 0.5, \delta = 0.6, q = 0.01$. For this set of parametric values, the critical value of $E$ has been calculated as 0.014. From this figure, it is seen that as the value of $E$ increases, then the system remains unstable within $0.01 \leq E \leq 0.014$, but the system continues stable steady state behaviour for $E > 0.014$. It can be concluded that the harvesting of Fish species may be responsible for the stability of the system.
Figure 3: Bifurcation diagrams as a function of harvesting rate of Fish species ($E$) with respect to population densities.

Using the set of parametric values $r = 2.0, k = 120, \beta = 0.6, \alpha = 4.0, \gamma = 0.5, \beta_1 = 0.4, d = 0.03654, \rho = 0.1, s = 0.2, s_1 = 0.2, \delta = 0.6, q = 0.01, E = 0.4$, bifurcation diagram of system (1) has been presented in Figure 4. The critical value of $\gamma_1$ has been obtained as 0.00604. From this figure, it is observed that the system remains unstable or continues the oscillatory behaviour for $\gamma_1 < 0.00604$, but the system continues stable steady state behaviour for $\gamma_1 \geq 0.00604$. So, it can be concluded that due to availability of food for Fish may be stabilized the system.
Now the numerical solution of the optimal harvesting problem has been solved by using
the set of parametric values: \( r = 2.0, k = 100.0, \beta = 0.6, \alpha = 1.0, \gamma = 0.7, \beta_1 = 0.5, d = 0.345, \rho = 0.2, \gamma_1 = 0.6, s = 0.5, s_1 = 0.4, \delta = 0.01, q = 0.02 \). First we solve the state
equations (1) by using Forward Runge-Kutta method within a specified time interval. Next,
we solve the co-state equations (5)-(7) by using Backward Runge-Kutta method to solve
the optimal harvesting problem (4). Then the optimal harvesting results are shown in the
following figures with respect to selling price of Fish (\( p \)), the cost of harvesting (\( c \)) and
instantaneous annual discount rate (\( \delta_1 \)) respectively. From Figure 5, it is observed that as
the selling price of unit biomass of Fish increases, then the optimal harvesting rate of Fish
species gradually increases. From Figure 6, it is observed that as the cost of harvesting of
Fish increases, then the optimal harvesting of Fish first, gradually decreases after that it
goes to the equilibrium level. From Figure 7, it is seen that as the annual discount rate of
selling price increases, then the optimal harvesting of Fish gradually increases. So, it can be
concluded that the increase of annual discount rate of selling price can increase the optimal
harvesting rate of Fish.
Figure 5: Optimal harvesting of Fish species with respect to selling price \((p)\).

Figure 6: Optimal harvesting of Fish species with respect to harvesting cost \((c)\).
Figure 7: Optimal harvesting of Fish species with respect to annual discount rate ($\delta_1$) of Fish production cost.

10 Conclusion

In this paper, a three species predator-prey model among the interaction of Phytoplankton, Zooplankton and Fish has been developed. Logistic growth of Phytoplankton has been considered in the absence of Zooplankton and Fish. It is assumed that Phytoplankton is consumed by Zooplankton and Fish consumes Phytoplankton as well as Zooplankton. It is considered that consumption rate of Zooplankton by Fish is greater than the consumption rate of phytoplankton by Fish. That’s why two types of functional responses Holling I & II have been used here. Then boundedness of all solutions of the system (1) has been studied. The local stability of the system has been studied around each of the possible equilibrium points. Also, global stability of the interior equilibrium point has been investigated by using Lyapunov function. The existence condition of Hopf bifurcation has been studied with respect to the half saturation constant $\alpha$. Then the optimal harvesting rate of Fish species has been determined with the help of optimal control theory.

From our numerical simulation results, it is found that the half saturation constant ($\alpha$), the harvesting rate of Fish species ($E$) and the consumption rate of Zooplankton by Fish ($\gamma_1$) can change the system dynamics. The system loses its stability whenever $\alpha < 1.18$ and
the system remains stable if $\alpha > 1.18$ (See Figure 2). It can be concluded that due to the increase of half saturation constant (increase of capability of consumption of food), the system may go into the stable state behaviour. Also, it is found that the system continues oscillatory behaviour for $0.01 \leq E \leq 0.014$, but the system becomes stable for $E > 0.014$ (See Figure 3). So, it can be concluded that optimal harvesting of Fish may make our proposed system stable. It is observed that the system continues oscillatory behaviour for $\gamma_1 < 0.00604$, but the system continues stable steady state behaviour for $\gamma_1 \geq 0.00604$ (See Figure 4). So, it can be concluded that the system may become stable for the higher rate of consumption of Zooplankton by Fish species. From the numerical simulation results of the optimal harvesting problem, it is observed that as the selling price ($p$) of unit biomass of Fish (See Figure 5) and instantaneous annual discount rate ($\delta_1$) (See Figure 7) increases then the optimal harvesting rate of Fish gradually increases. So, it can be concluded that the harvesting of Fish species will be increased if the selling price of Fish and the annual discount of Fish production cost increases. It is also found that the optimal harvesting rate of Fish decreases due to the increase of cost of harvesting ($c$) of Fish (See Figure 6). Finally, it can be said that the model can be implemented in a Fishery system. Several research works are going on Phytoplankton-Zooplankton-Fish species biological dynamics. In the near future, we shall modify this model, including some biological factors such as intra-species competition on Fish species, linear harvesting of Zooplankton species, supplying of additional food to the Fish, effects of toxicants released by Phytoplankton on Fish species etc.

References


Existence and Extremal Solution of Boundary Value Problem for Nonlinear Hybrid Fractional Differential Equation in Banach Algebras

B.D. Karande & Pravin M. More

Department of Mathematics, Maharashtra Udayagiri Mahavidyalaya Udgir, India
Department of Mathematics, B.S.SASC College Makni, India

Email: pravinmorepravin@gmail.com

Corresponding Author: Pravin M. More

Abstract: In this work we study the existence and extremal solution for the boundary value problem of the nonlinear hybrid fractional differential equation by using hybrid fixed point theorem in Banach Algebra due to Dhage’s theorem.

Keywords: Banach algebra, Hybrid fixed point theorem, Existence result, Locally attractive solution, Extremal solution

1. Introduction

The concept of fractional calculus has gained considerable popularity and importance during the past ten decades and has numerous applications in diverse fields of science and engineering. The credit of theory of fractional derivatives goes to L’Hospital (1661-1704) in denotation of the derivative of order \(1/2\). The theory of derivative and integral of arbitrary order in subsequent mentioned of fractional derivatives was made in same context by Euler (1730), Lagrange (1772), Lacriox (1819), Fourier (1822), Laplace (1832) , Liouville (1832), Riemann (1847), Greer (1859) , Holmgren (1865), Grunwald (1867), Letnikov(1868), Sonin (1869), Laurent (1884), Nekrassov(1888), Krug (1890) and Wely (1917) etc. Recently there are several works on fractional derivatives and integral equations, for instances, we refer a few (see. [1],[8],[9],[13]). The theory of fractional derivative and integral are very suitable for solving linear and nonlinear functional equations.

Bhi and Lu [1] and Zhao and Wang [18] studied the following two point boundary value problem of fractional differential equation

\[
D^\alpha_0 u(t) + f(t, u(t)) = 0 \quad 0 < t < 1, \quad u(0) = u(1) = 0
\]

where \(1 < \alpha \leq 2\) is a real number and \(D^\alpha_0\) is the standard Riemann-Liouville fractional derivative. By the properties of Green function, they have some multiple positive solutions for singular and nonsingular boundary value problem by mean of Lery-Schauder fixed point theorem on cone. They obtained the existence of positive solutions by using Guo-Krasnosel’kii fixed point theorem and Leggett-Williams fixed point theorem. In the recent years, quadratic perturbations of nonlinear differential equations have attracted much attention to many researchers. We call such differential equations as hybrid differential equations and have been studied in some works, for instances: [2],[11],[12],[14],[15] and [17].

Zhao et al [3] studied first order hybrid differential equations (In shorts FHDE) involving Riemann-Liouville differential operators of order \(0 < q < 1\),

\[
\frac{d}{dt} \left[ \frac{x(t)}{f(t,x(t))} \right] = g(t, x(t)), t \in [0,1],
\]

\[
x(0) = x(1) = 0
\]

where \(f \in C([0,T] \times \mathbb{R}, \mathbb{R}\setminus\{0\})\) and \(g \in C([0,T] \times \mathbb{R}, \mathbb{R})\), was studied in the works [2],[5]. They established the existence and uniqueness results for fractional hybrid differential equations and initiated to study of such systems.
In this paper we study the existence and extremal solution with the boundary value problem for the following nonlinear hybrid fractional differential equation (NHFDE)

\[ D_0^\alpha \left[ \frac{x(t)}{f(t,x(t))} \right] + g(t,x(t)) = 0, \quad 0 < t < 1 \quad (1.1) \]

\[ x(0) = x(1) = x'(0) = 0, \quad (1.2) \]

where \( 1 < \alpha \leq 2 \) is a real number and \( D_0^\alpha \) is the standard Riemann Liouville fractional derivative. The result has been obtained an existence and extremal solutions for the boundary value problem for above equation (1.1) by using hybrid fixed point theorem, two operators in Banach Algebra due to Dhage.

Let \( \mathbb{R} \) be the real line and \( J = [0,1] \) be a bounded interval in \( \mathbb{R} \). Let \( C(J \times \mathbb{R}_+, \mathbb{R}) \) denote the class of continuous functions \( f: J \times \mathbb{R}_+ \rightarrow \mathbb{R} \) and let \( C(J \times \mathbb{R}_+, \mathbb{R}) \) denote the class of functions \( g: J \times \mathbb{R}_+ \rightarrow \mathbb{R} \) such that

i. the map \( t \mapsto g(t,x) \) is measurable for each \( x \in \mathbb{R} \), and

ii. the map \( x \mapsto g(t,x) \) is continuous for each \( t \in J \).

The class \( C(J \times \mathbb{R}_+, \mathbb{R}) \) is called the Caratheodory class of functions on \( J \times \mathbb{R} \) which are Lebesgue integrable when bounded by a Lebesgue integrable function on \( J \).

In this paper, we assume \( f \in C([0,1] \times \mathbb{R}, \{0\}) \) and \( g \in C([0,1] \times \mathbb{R}_+, \mathbb{R}) \).

In section 2 we shall give some definitions and Lemmas to prove the main theorems. Section 3 establishes the existence theory. In section 4, we establish the existence of solution of boundary value problem of equation (1.1) and (1.2) by using hybrid fixed point theorem in Banach algebras due to Dhage. Section 5 deals with solving the existence of extremal solutions. Finally we conclude by illustrating the main result by two examples in section 6.

2. Preliminaries

We recall some basic definitions, lemmas and known results for fractional calculus and functional analysis to prove the main theorems, which can be found in the works; see [1], [2], [4], [7], and [8].

**Definition 2.1 [12]**. The Riemann-Lowville fractional derivative of order \( \alpha > 0 \) of a function \( f: (0, +\infty) \rightarrow \mathbb{R} \) is given by

\[ D_0^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \, ds, \]

where \( n = [\alpha] + 1 \), \([\alpha]\) denotes the integer part of number \( \alpha \), provided that the right side is point wise defined on \((0, +\infty)\).

**Definition 2.2[12]**. The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( f: (0, +\infty) \rightarrow \mathbb{R} \) is given by

\[ I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \]

provided that the right side is point wise defined on \((0, +\infty)\).

From the definition of the Riemann-Liouville derivative, we can obtain the following statements.

**Definition 2.3 [2]**. A mapping \( \phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is called dominating function or in short \( D \)-function if it is continuous and monotonic non-decreasing function satisfies \( \phi(0) = 0 \).

**Definition 2.4 [2]**. Let \( X \) be the Banach algebra with norm. A mapping \( G: E \rightarrow E \) is called \( D - \)Lipschitz or nonlinear \( D - \)Lipschitz, if there exists a continuous non-decreasing function \( \phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that
\[ \|Gx - Gy\| \leq \|x - y\|, \text{ for all } x, y \in X, \text{ where } \|\Phi(0)\| = 0. \]

If \( \Phi(r) = kr, \quad k > 0, \) then \( k \) is called \( G \)-Lipschitz constant.

If \( k < 1, \) \( G \) is called contraction with contraction constant \( k. \)

Finally \( G \) is called nonlinear \( D \)-contraction if it is nonlinear \( D \)-Lipschitz with \( \Phi(r) < r \) with \( r > 0. \)

**Lemma 2.1**[18]. Let \( \alpha > 0. \) Assume that \( u \in C(0,1) \cap L(0,1), \) then the fractional differential equation \( D_0^\alpha u(t) = 0 \) has a unique solution

\[ u(t) = c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \cdots + c_n t^{\alpha - n}, c_i \in \mathbb{R}, i = 1, 2, \ldots, n; \]

where \( n \) is the smallest integer greater than or equal to \( \alpha. \)

**Lemma 2.2**[18]. Assume that \( u \in C(0,1) \cap L(0,1) \) with a fractional derivative of order \( \alpha > 0 \) that belongs to \( C \in C(0,1) \cap L(0,1). \) Then

\[ I_0^\alpha D_0^\alpha u(t) = u(t) + c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \cdots + c_n t^{\alpha - n}, \]

for some \( c_i \in \mathbb{R}, \) where \( i = 1, 2, \ldots, n, n = [\alpha] + 1, \) and \( n \) is the smallest integer greater than or equal to \( \alpha. \)

In the following, we present the Green function of fractional differential equation with boundary value problem.

**Lemma 2.3**[18]. Let \( y \in C[0,1] \) and \( 1 < \alpha \leq 2. \) The unique solution of problem

\[
\begin{align*}
D_0^\alpha \left[ \frac{x(t)}{f(t,x(t))} \right] + g(t,x(t)) &= 0, \quad 0 < t < 1 \\
x(0) &= x(1) = 0
\end{align*}
\]

is

\[ x(t) = f(t,x(t)) \int_0^1 G(t,s)y(s)ds \]

where \( G(t,s) = \begin{cases} 
\frac{[\Gamma(\alpha)]^{\alpha-1}(t-s)^{\alpha-1}}{[\Gamma(\alpha)]^{\alpha}}, & 0 \leq t \leq s \leq 1 \\
\frac{[\Gamma(\alpha)]^{\alpha-1}(t-s)^{\alpha-1}}{[\Gamma(\alpha)]^{\alpha-1}}, & 0 \leq s \leq t \leq 1
\end{cases} \]

Here \( G(t,s) \) is called the Green function of boundary value problem (2.1) and (2.2).

We apply hybrid fixed point theorem due to Dhage [8] for proving the existence result.

**Lemma 2.5**[4]. Let \( S \) be a non-empty, closed convex and bounded subset of the Banach algebra \( X \) and let \( A:X \to X \) and \( B:S \to X \) be two operators such that

(a) \( A \) is Lipschitzian with a Lipschitz constant \( \alpha, \)

(b) \( B \) is completely continuous,

(c) \( x = AxBy \Rightarrow x \in S \forall y \in S, \) and

(d) \( \alpha M < 1, \) where \( M = \|B(s)\| = \sup\{\|B(x)\| : x \in S\}. \)

Then the operator equation \( AxBy = x \) has a solution in \( S. \)

3. Existence Theory

In this section, we prove the existence result for fractional differential equation (1.1) on \( J = [0,1] \) under the Lipschitz and Caratheodory conditions on nonlinearities involved in it. We place the fractional differential equation (1.1) in the space \( BC(J, \mathbb{R}_+) \) of continuous real-valued function defined on \( J. \) For convenience, we denote \( T = \int_0^1 G(s,s)ds. \)
Lemma 3.1[14]. The function $G(t, s)$ defined by (2.3) satisfies the following conditions:

1) $G(t, s) = G(1 - s, 1 - t)$, for $t, s \in (0, 1)$;
2) $t^{a-1}(1 - s)(1 - s)^{a-1} \leq \Gamma(a), G(t, s) \leq (a - 1)s(1 - s)^{a-1}$, for $t, s \in (0, 1)$;
3) $G(t, s) > 0$, for $t, s \in (0, 1)$; and
4) $t^{a-1}(1 - s)(1 - s)^{a-1} \leq \Gamma(a), G(t, s) \leq (a - 1)s(1 - s)^{a-1}$, for $t, s \in (0, 1)$.

Let $X = BC(J, \mathbb{R}_+)$ denotes the function space of all continuous real valued function on $J$. Define $\|\cdot\|$ and multiplication on $E$ by

\[
\|x\| = \sup_{t \in J} |x(t)| \quad (3.1)
\]
\[
(x, y)(t) = x(t)y(t), \forall t \in J. \quad (3.2)
\]

Let $X = BC(R_+, \mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and let $\Omega$ be the subset of $X$. Let a mapping $G: X \to X$ be an operator and consider the following equations in $X$ namely, $x(t) = (Gx)(t) \forall t \in \mathbb{R}_+$. Clearly $X = BC(\mathbb{R}_+, \mathbb{R})$ becomes a Banach space with respect to above norm and multiplication. By $L^1(\mathbb{R}_+)$ we denote the space of Lebesgue integrable function defined on $J$ with the norm $\|\cdot\|_{L^1}$ defined by

\[
\|h\|_{L^1} = \int_{0}^{T} |G(s)| \, ds \quad (3.3)
\]

Denote $L^1(a, b)$ as the Lebesgue integrable function on interval $(a, b)$. Let $x \in L^1(a, b)$ and $\alpha > 0$ be a fixed number.

**Definition 3.1 [6]:** A Mapping $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if;

i) $t \to g(t, x)$ is measurable for each $x \in \mathbb{R}$ and

ii) $t \to g(t, x)$ is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore Caratheodory function $g$ is $L^1$ – Caratheodory if

iii) For each real number $r > 0$ there exist a function $h_r \in L^1(\mathbb{R}_+, \mathbb{R})$ such that

\[
|g(t, x)| \leq h_r(t) \ i.e. \ t \in \mathbb{R}_+ \ \forall x \in \mathbb{R} \ \text{with} \ |x| \leq r.
\]

Finally a Caratheodory function $g$ is $L^1_X$ – Caratheodory if;

iv) There exists a function $h \in L^1(\mathbb{R}_+, \mathbb{R})$ such that

\[
|g(t, x)| \leq h(t) \ i.e. \ t \in \mathbb{R}_+ \ \forall x \in \mathbb{R}.
\]

For convenience, the function $h$ is referred as a bound function for $g$.

**4. Main Results**

We need the following hypothesis for existence of solution of NHFDE.

(H1): the function $f: J \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded with bound $F = \sup_{(t, x) \in J} |f(t, x)|$;

(H2): there exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|, \forall t \in J$ and $x, y \in \mathbb{R}$;

(H3) there exists a function $h \in L^1(J, \mathbb{R}_+)$ such that $|g(t, x)| \leq h(t), t \in J, \forall x \in \mathbb{R}_+$; and

(H4): the function $v: \mathbb{R}_+ \to \mathbb{R}$ defined by the formula
\[ v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-a}} \, ds = T\|h\|_{L^1} \] is bounded on \( \mathbb{R}_+ \) and vanish at infinity, for all \( x \in \mathbb{R}_+ \).

**Remark 4.1:** If the hypothesis \((H_3)\) and \((H_4)\) hold, then there exists a constant \( k_1 > 0 \) such that

\[ k_1 = \sup_{t \geq 0} \frac{v(t)}{\Gamma(\alpha)} : t \in \mathbb{R}_+. \]

**Theorem 4.1.** Assume that the hypotheses \((H_1)\) and \((H_4)\) hold. Further, if

\[ \text{LT}\|h\|_{L^1} < 1, \quad (4.1) \]

Then the boundary value problems (1.1) and (1.2) has a solution in the space \( BC(\mathbb{R}_+, \mathbb{R})\) and are locally attractive on \( \mathbb{R}_+ \).

**Proof.** Set \( X = BC(j, \mathbb{R}) \) and a subset \( S \) of \( X \) defined by

\[ s = \{ x \in X \mid \|x\| \leq N \}, \quad (4.2) \]

where \( N = \frac{f_0 \|h\|_{L^1}}{1 - \text{LT}\|h\|_{L^1}} \) and \( f_0 = \sup_{t \in J} |f(t, 0)| \).

Clearly \( S \) is a closed, convex and bounded subset of the Banach space \( X \). In view of Lemma (2.3), boundary value problem (1.1) and (1.2) is equivalent to the equation

\[ x(t) = f(t, x(t)) \int_0^1 G(t, s) g(s, x(s)) ds, \quad t \in J. \quad (4.3) \]

Define two operators

\[ Ax(t) = f(t, x(t)), \quad t \in J. \quad (4.4) \]

and \( Bx(t) = \int_0^1 G(t, s) g(s, x(s)) ds, \quad t \in J. \quad (4.5) \]

Then equation (3.3) is transformed into the operator equation as

\[ Ax(t)Bx(t) = x(t), \quad t \in J. \quad (4.6) \]

We shall show that the operators \( A \) and \( B \) satisfy all the conditions of Lemma 2.5.

**Step I:** We show that \( A \) is Lipschitz on \( X = BC(\mathbb{R}_+, \mathbb{R}) \).

Let \( x, y \in X \), then the hypothesis \( H_1 \) is

\[ |Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq L\|x - y\|, \quad \forall t \in J. \]

Taking supremum over \( t \), we obtain

\[ \|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in X. \]

Thus \( A \) is Lipschitz on \( X \) with Lipschitz constant \( L \).

**Step II:** Now we show that \( B \) is compact operator on \( S \) into \( X \). First we show that \( B \) is continuous on \( S \).

Let \( \{x_n\} \) be a sequence in \( S \) converging to a point \( x \in S \). Then by Lebesgue dominated convergence theorem, for \( t \in J \), we obtain

\[
\lim_{n \to \infty} Bx_n(t) = \lim_{n \to \infty} \int_0^1 G(t, s) g(s, x_n(s)) ds = \int_0^1 G(t, s) \lim_{n \to \infty} g(s, x_n(s)) ds = \int_0^1 G(t, s) g(s, x(t)) ds = Bx(t), \quad \forall t \in J.
\]
Taking supremum over $t$, shows that $B$ is continuous operator on $X$.

Sep III: Next we show that $B$ is a compact operator on $X$. For this, it is enough to show that $B(S)$ is uniformly bounded and equi-continuous set on $X$. Let $x \in S$ be an arbitrary. Then by hypothesis $(H_1)$,

$$\|Bx\| = \left| \int_0^1 G(t,s)g(s,x(s))ds \right| \leq \int_0^1 G(t,s)\|g(s,x(s))\|ds \leq \int_0^1 G(s,s)h(s)ds$$

$$\leq T\|h\|_{L^1} = v(t), \forall t \in I \text{ and } \forall x \in S.$$

This shows that $B$ is uniformly bounded on $S$.

On the other hand, let $t_1, t_2 \in I$ with $t_1 < t_2$, then for any $x \in S$, we have

$$|Bx(t_2) - Bx(t_1)| = \left| \int_0^1 G(t_2,s)g(s,x(s))ds - \int_0^1 G(t_2,s)g(s,x(s))ds \right|$$

$$\leq \|h\|_{L^1} \left| \int_0^1 \frac{t_2(s(1-s))^{\alpha-1}(1-t_2^{1-s})^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t_2} \frac{t_1^{1-s}s^{\alpha-1}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds - \int_0^{t_1} \frac{t_1^{1-s}s^{\alpha-1}(1-t_2^{1-s})^{\alpha-1}}{\Gamma(\alpha)} ds \right|$$

$$\leq \|h\|_{L^1} \int_0^1 \frac{t_2(s(1-s))^{\alpha-1}(1-t_2^{1-s})^{\alpha-1}}{\Gamma(\alpha)} ds + \|h\|_{L^1} \int_0^{t_2} \frac{t_1^{1-s}s^{\alpha-1}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$\leq \|h\|_{L^1} \left[ t_2^{\ominus_{\alpha-1}} - t_1^{\ominus_{\alpha-1}} + t_2^{\ominus_{\alpha-1}} - t_2^{\ominus_{\alpha-1}} \right]$$

$$\leq T\|h\|_{L^1} [t_2^{\ominus_{\alpha-1}} - t_1^{\ominus_{\alpha-1}}] + (t_2 - t_1)$$

Thus the right hand of above inequality does not depend on $X$ and tends to zero.

Therefore $|Bx(t_2) - Bx(t_1)| \to 0$ as $t_1 \to t_2$. Hence for $t_1, t_2 \geq T$ we have

$$|Bx(t_2) - Bx(t_1)| = \left| t_2 \int_0^1 G(t_2,s)g(s,x_n(s))ds - \int_0^{t_1} G(t_2,s)g(s,x_n(s))ds \right|$$

$$\leq \left| \int_0^{t_2} h(s)ds - \int_0^{t_1} h(s)ds \right| \leq \left| \int_0^{t_2} h(s)ds \right| + \left| \int_0^{t_1} h(s)ds \right|$$

$$\leq v(t_2) + v(t_1)$$

$$\leq \frac{\alpha}{2} + \frac{\alpha}{2} \leq \alpha \text{ as } t_1 \to t_2.$$

Hence $\forall t_1, t_2 \in \mathbb{R}_+$ we have

$$|Bx_n(t_2) - Bx_n(t_1)| \leq |Bx_n(t_2) - Bx_n(T)| + |Bx_n(T) - Bx_n(t_1)|$$

If $t_1 \to t_2, t_1 \to T$ and $T \to t_2$ then

$$|Bx_n(t_2) - Bx_n(t_1)| \to 0, \ |Bx_n(t_2) - Bx_n(T)| \to 0 \text{ and } |Bx_n(T) - Bx_n(t_1)| \to 0.$$

Hence $\{Bx_n\}$ is an equicontinuous sequence in $X = BC(\mathbb{R}_+ \mathbb{R})$ and is relatively compact by the Arzela-Ascoli Theorem. As a result, $B$ is completely continuous operator on $S$.

Step IV: We show that the hypothesis (c) of Lemma 2.5 is satisfied.

Let $x \in X$ and $y \in Y$ be arbitrary such that $x = AxBy$ then by assumptions $(H_1 - H_3)$, we have
\( |x(t)| = |Ax(t)||By(t)| = |f(t,x(t))|\left|\int_0^1 G(t,s) g(s,y(s)) \, ds\right|
\leq \left|\int f(t,x(t)) - f(t,0) + |f(t,0)|\right|\left|\int_0^1 G(s,s) g(s,y(s)) \, ds\right|
\leq [L|x(t)| + F_0]|T|h_{L^1}
Thus \( |x(t)| \leq \frac{F_0|h_{L^1}}{1-LT|h_{L^1}} \)

Taking supremum over, \( t \in \mathbb{R}_+ \), we obtain \( \|x\| \leq \frac{F_0|h_{L^1}}{1-LT|h_{L^1}} = N. \)

This shows that the hypothesis of Lemma (2.5) is satisfied. Finally, we have

Step V: \( M = \|B(S)\| = \sup \{\|Bx\| : x \in S\} \)
\leq \sup \left\{\sup_{t \in J} \left|\int_0^1 G(t,s) g(s,x(s)) \, ds\right| : x \in S\right\} \leq \sup \left\{\sup_{t \in J} \left|\int_0^1 h(s) \, ds\right| : x \in S\right\}
\leq \sup \left\{\sup_{t \in J} [T|h_{L^1}] : x \in S\right\}
\leq K_1

Therefore \( MK = LK_1 < 1 \). Thus the hypothesis (d) of Lemma 2.5 is satisfied.

Hence all the conditions of Lemma 2.5 are satisfied and hence the operator equation \( x = AxBy \) has a solution in \( S \). As a result, the boundary value problem (1.1) and (1.2) has a solution defined on \( J \). This completes the proof.

**Remark 4.1:** For the special case \( f(t,x) = 1 \), we can find the corresponding existence result in Bai and Lu (see [1]).

Step VI: Finally we show the locally attractivity of solution for NHFDE (1.1) and (1.2).

Let \( x \) and \( y \) be two solutions of equation (1.1) and (1.2) in \( X \) defined on \( \mathbb{R}_+ \). Then we have

\[ |x(t) - y(t)| = \left|\int f(t,x(t)) \left|\int_0^1 G(t,s) g(s,x(s)) \, ds\right| + f(t,y(t)) \left|\int_0^1 G(t,s) g(s,y(s)) \, ds\right|\right|\]
\leq \left|\int f(t,x(t)) \left|\int_0^1 G(t,s) g(s,x(s)) \, ds\right| \right| + \left|\int f(t,y(t)) \left|\int_0^1 G(t,s) g(s,y(s)) \, ds\right| \right| \, ds
\leq F \left\{\int_0^1 G(s,s) h(s) \, ds\right\} + F \left\{\int_0^1 G(s,s) h(s) \, ds\right\}
\leq 2F \left\{\int_0^1 G(s,s) h(s) \, ds\right\}
\leq 2F[T|h_{L^1}]
\leq 2F[v(t)] \quad [\because T|h_{L^1} = v(t)]

Since \( \lim_{t \to \infty} v(t) = 0 \), so that \( \lim_{t \to \infty} |x(t) - y(t)| = 0 \). This completes the proof.

**5.1 Existence of the Extremal Solution of NHFDE (1.1)**

A nonempty closed set \( K \) in Banach Algebra \( X \) is called a cone with vertex 0 if

i) \( K + K \subseteq K \), ii) \( \lambda K \subseteq K \) for \( \lambda \in \mathbb{R}_+ \), \( \lambda \geq 0 \), iii) \( (-K) \cap K = 0 \), where 0 is the element of \( X \).

A cone \( K \) is said to be positive of \( K \circ K \subseteq K \), where \( \circ \) is a multiplication composition in \( X \).

We introduce an order relation \( \leq \) in \( X \) as follows.
Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone $K$ is said to be normal if the norm $\|\|_K$ is semi-monotone increasing on $K$, that is there is a constant $N > 0$ such that

$$\|x\| \leq N\|y\|, \forall x, y \in K \text{ with } x \leq y.$$ 

It is known that if the cone $K$ is normal in $X$, then every order bounded set in $X$ is norm-bounded. These concepts appear in the works of Heikkila and Lakshmikantham (see, [3], [10]).

**Lemma 5.1 [6]:** Let $K$ be a positive cone in a real Banach Algebra $X$ and let $u_1, u_2, v_1, v_2 \in K$ be such that

$$u_1 \leq v_1, u_2 \leq v_2 \text{ then } u_1 u_2 \leq v_1 v_2.$$ 

We use the following fixed point theorem due to Dhage [5] for proving the existence of extremal solution for BVP (2.1) under certain monotonically conditions.

**Lemma 5.2[6]:** Let $K$ be a cone in Banach Algebra $X$ and let $a(x) \leq b(x)$ be such that $a \leq b$. Suppose that $A, B: [a, b] \to K$ are two nondecreasing operators such that

a) $A$ is Lipschitzian with a Lipschitz constant $\alpha$.

b) $B$ is complete.

c) $Ax Bx \in [a, b] \forall x \in [a, b].$

Further, if the cone $K$ is positive and normal, then the operator equation $Ax Bx = x$ has a least and a greatest positive solution in $[a, b]$ whenever $aM < 1$, where $M = \|B(s)\| = \sup\{\|B(x)\| : x \in [a, b]\}$.

We equip the space $BC(J, \mathbb{R}_+)$ with order relation $\leq$ with the help of the cone $K$ defined by

$$K = \{x \in BC(J, \mathbb{R}) : x(t) \geq 0 \ \forall t \in J. \} \quad (5.1)$$

It is well known that cone $K$ is positive and normal in $BC(J, \mathbb{R}_+)$. 

**Definition 5.2 ([2], [16]):** A function $u \in BC(\mathbb{R}_+, \mathbb{R})$ is called a lower function of NHFDE on $\mathbb{R}_+$ if the function $t \to \frac{u(t)}{f(t, u(t))}$ is continuous and

$$D_0^\alpha \left[\frac{u(t)}{f(t, u(t))}\right] + g(t, u(t)) = 0, \quad 0 < t < 1$$

$$x(0) = x(1) = 0.$$ 

Again a function $v \in BC(\mathbb{R}_+, \mathbb{R})$ is called an upper function of NHFDE on $\mathbb{R}_+$ if the function $t \to \frac{v(t)}{f(t, v(t))}$ is continuous and a function $u \in BC(J, \mathbb{R})$ is a solution of NHFDE is lower as well as an upper solution of the NHFDE defined on $J$.

We consider the following assumptions

(B$_1$) $f: J \times \mathbb{R} \to \mathbb{R}_+, g: J \times \mathbb{R} \to \mathbb{R}_+.$

(B$_2$) The NHFDE (2.1) has lower solution $a$ and an upper solution $b$ defined on $J$ with $a < b$.

(B$_3$) The function $x \to \frac{x}{f(t, x)}$ is increasing in the interval $[\min_{t \in J} a(t), \max_{t \in J} b(t)]$ almost on $J$ with $a < b$.

(B$_4$) The function $f(t, x)$ and $g(t, x)$ are nondecreasing in $x$ almost everywhere for $t \in J$.

(B$_5$) There exists a function $K \in L^1(J, \mathbb{R})$ such that $g(t, b(t)) \leq K(t)$.

In particular, the hypothesis (B$_5$) holds if $f$ is continuous and $g$ is $L^1 -$ Caratheodory on $J \times \mathbb{R}$.

**Theorem 5.1** Suppose that the assumptions (H$_2$) and (B$_1$ $-$ B$_3$) hold. Further, if $T\|h\|_{L^1} < 1 \quad (5.2)$

then NHFDE (2.1) has minimal and maximal positive solution defined on $J$.

**Proof:**

Now NHFDE (2.1) is equivalent to the integral equation (2.5) defined on $J$.

Let $X = BC(J, \mathbb{R})$. Define the two operators $A$ and $B$ on $X$ by the equations (4.5) and (4.6) Then the integral equation is transformed into an operator equation $Ax Bx(t) = x(t)$ in Banach Algebra $X$. 30
The hypothesis (B1) implies $A, B : [a, b] \to K$. Since the cone $K$ in $X$ is normal, $[a, b]$ is normal-bounded set in $X$. By the proof of Theorem 4.1, $A$ is Lipschitzian with Lipschitz constant $L$ and $B$ is completely continuous operator on $[a, b]$. Again hypothesis (B3) implies that $A$ and $B$ are non-decreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ such that $x \leq y$. Then by hypothesis (B3), we have

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t) \quad \forall t \in J.$$ 

Similarly, we have

$$Bx(t) = \int_{0}^{1} G(t, s) g(s, x(s))ds \leq \int_{0}^{1} G(t, s)g(s, y(s))ds = By(t) \quad \forall t \in J.$$ 

So $A$ and $B$ are non-decreasing operators on $[a, b]$. By Lemma 5.1 and hypothesis (B4) together imply that

$$a(t) \leq f(t, a(t)) \int_{0}^{1} G(t, s) g(s, a(s))ds \leq f(t, x(t)) \int_{0}^{1} G(t, s) g(s, x(s))ds$$

$$\leq f(t, b(t)) \int_{0}^{1} G(t, s) g(s, b(s))ds$$

$$\leq b(t), \quad \forall t \in \mathbb{R}_+ \text{ and } x \in [a, b].$$

So $a(t) \leq Ax(t) \leq b(t)$ $\forall t \in \mathbb{R}_+$ and $x \in [a, b]$. Hence $Ax \in [a, b]$ $\forall x \in [a, b]$.

Again $M = \|B([a, b])\| = \sup \{|Bx| : x \in [a, b]\}$

$$\leq \sup \left\{ \sup_{t \in J} \int_{0}^{1} |g(s, x(s))ds| : x \in [a, b] \right\}$$

$$\leq T\|h\|_{L^1}$$

and so $aM \leq T\|h\|_{L^1} < 1$.

Now we apply Lemma 5.2 for the operator $Ax = x$ to yield NHFDE (2.1) to show minimal and maximal positive solution in $\mathbb{R}_+$. This completes the proof.

**5.2 Examples**

**Example 5.1:** Consider the boundary value problem

$$D_{0}^{\frac{3}{2}}x(t) + \sin x = 0, \quad 0 < t < 1,$$

$$x(0) = x(1) = 0$$

(5.3)

(5.4)

Let $f(t, x(t)) = 1, g(t, x) = \sin x, h(t) = 1$. Then for $\alpha = \frac{3}{2}$, hypothesis (H2) and (H3) hold, since

$$T = \int_{0}^{\frac{1}{2}} [\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}]ds = \int_{0}^{\frac{1}{2}} \frac{(1-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}ds = \frac{\sqrt{\pi}}{4}.$$ 

Choosing $L=1$ then $T\|h\|_{L^1} = \frac{\sqrt{\pi}}{4} < 1$. Therefore boundary value problem (5.3) and (5.4) has a solution.

**Example 5.2:** Consider the boundary value problem for $\alpha = \frac{3}{2}$

$$D_{0}^{\alpha} \left[ \frac{x(t)}{\sin x + 2} \right] + \cos x = 0, \quad 0 < t < 1$$

$$x(0) = x(1) = 0$$

(5.5)

(5.6)

Let $f(t, x) = \sin x + 2, g(t, x) = \cos x, h(t) = 1$. Then the hypothesis (H2) and (H3) hold.

Since $T = \frac{\sqrt{\pi}}{4}$, choosing $L = 1$ then $T\|h\|_{L^1} = \frac{\sqrt{\pi}}{4} < 1$.

Therefore the boundary value problem (5.5) and (5.6) has a solution.

**Conclusion**

In this work we have studied the existence solution for boundary value problem for nonlinear hybrid fractional differential equation. This topic may be applicable in numerous fields of science and engineering and currently it is a central point to the researchers.
Acknowledgements
The authors are thankful to the referees for their constructive suggestions for the improvement of the content of the paper.

References
A New Two - Parameter Lindley Distribution

Rama Shanker & Umme Habibah Rahman
Department of Statistics, Assam University, Silchar, India
Email: shankerrama2009@gmail.com; umme.habibah.rahman17@gmail.com
Corresponding Author: Rama Shanker

Abstract: In this paper, a new two - parameter Lindley distribution has been proposed. Descriptive statistical properties along with order statistics, Fisher information matrix and confidence interval of the proposed distribution have been discussed. Parameters are estimated by the method of Maximum Likelihood estimation. A real lifetime data has been presented to test the goodness of fit of the proposed distribution over other one parameter and two –parameter Lindley family of distributions.

Keywords: Lindley distribution, Two-parameter Lindley distributions, Statistical properties, Maximum likelihood estimation, Fisher information matrix.

1. Introduction

The exponential distribution and the Lindley distribution were the two classical lifetime distributions for modeling lifetime data. Lindley [5] introduced a lifetime distribution known as Lindley distribution defined by the probability density function (pdf) and cumulative distribution function (cdf)

\[ f(x;\theta) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}; \quad x > 0 \text{ and } \theta > 0 \]  
\[ F(x;\theta) = 1 - \left[ 1 + \frac{\theta x}{\theta + 1} \right]e^{-\theta x}; \quad x > 0 \text{ and } \theta > 0 \]

Lindley distribution, being a convex combination of exponential and gamma distribution, gives better fit than exponential distribution and it is more flexible than the exponential distribution. Ghitany et al [3] have studied many interesting properties, estimation of parameter using both the method of moments and the method of maximum likelihood, and application of Lindley distribution. Recently several two-parameter Lindley distributions have been introduced by different researchers. Some important two-parameter lifetime distribution proposed by different researchers are presented in the following Table 1.

<table>
<thead>
<tr>
<th>Name of the distributions</th>
<th>probability density function (pdf)</th>
<th>Introducers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-parameter Lindley distribution-1 (TPLD-1)</td>
<td>[ f(x;\theta,\alpha) = \frac{\theta^2}{\theta \alpha + 1}(\alpha + x)e^{-\theta x}; x &gt; 0, \theta &gt; 0, \theta \alpha &gt; -1 ]</td>
<td>Shanker and Mishra (2013)[8]</td>
</tr>
</tbody>
</table>
### Two-parameter Lindley distribution - 2 (TPLD-II)

\[ f(x; \theta, \alpha) = \frac{\theta^2}{\theta + \alpha} (1 + \alpha x) e^{-\theta x}; x > 0, \theta > 0, \alpha > 0 \]

*Shanker et al (2013)[7]*

### Quasi Lindley distribution (QLD)

\[ f(x; \theta, \alpha) = \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x}; x > 0, \theta > 0, \alpha > -1 \]

*Shanker and Mishra (2013)[9]*

### New Quasi Lindley distribution (NQLD)

\[ f(x; \theta, \alpha) = \frac{\theta^2}{\theta^2 + \alpha} (\alpha + \theta x) e^{-\theta x}; x > 0, \theta > 0, \alpha > 0 \]

*Shanker and Amanuel (2013)[10]*

The statistical properties, estimation of parameters using both the method of moments and maximum likelihood estimation and applications of these two-parameter Lindley distributions are available in the respective papers. Some of the important generalizations and extensions of Lindley distribution have been done by researchers including Bakouch et al [2] and Ghitany et al [4], Nadarajah et al [6], Zakerzadeh and Dolati [11], some among others.

The main purpose of this paper is to introduce a new two-parameter Lindley distribution which gives much better fit than the existing two-parameter Lindley distributions in statistics literature. Several properties like reliability analysis, mean residual life function and moments have been discussed. Estimation of parameters is estimated by the method of maximum likelihood and Fisher information matrix with confidence interval is also given. At last, goodness of fit of the proposed distribution and its comparative fit with other one and two-parameter Lindley family of distributions are discussed.

### 2. A New Two-Parameter Lindley Distribution

The pdf and the cdf of the new two-parameter Lindley distribution (NTPLD) can be expressed as

\[ f(x; \alpha, \theta) = \frac{\theta^{\alpha+1} (\alpha + x^\alpha) e^{-\theta x}}{\alpha \theta^\alpha + \Gamma (\alpha)}; x > 0; \alpha > 0 \text{ and } \theta > 0, \quad (2.1) \]

\[ F(x; \alpha, \theta) = \frac{\alpha \theta^\alpha (1 - e^{-\theta x}) + \gamma (\alpha + 1, \theta x)}{\alpha \theta^\alpha + \Gamma (\alpha + 1)}; x > 0; \alpha > 0 \text{ and } \theta > 0. \quad (2.2) \]

where \( \alpha \) is the shape parameter and \( \theta \) is the scale parameter and \( \gamma (\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt \) is the lower incomplete gamma function. The NTPLD has a proper density function since

\[ \lim_{x \to \infty} F(x; \alpha, \theta) = 1 \quad \text{ and } \quad \lim_{x \to \infty} F(x; \alpha, \theta) = 0 \]

Graphs of the pdf and cdf of NTPLD has shown in figures 1 and 2 respectively for varying values of the parameters \( \alpha \) and \( \theta \).
3. Statistical Properties

In this section, statistical properties including asymptotic behavior, mean residual life function, reliability analysis of NTPLD have been studied.

3.1. Asymptotic Behavior

The asymptotic behavior of NTPLD for $x \to 0$ and $x \to \infty$ are

$$\lim_{x \to 0} f(x; \alpha, \theta) = \lim_{x \to \infty} \left[ \frac{\theta^{\alpha+1}(\alpha + x^\alpha)\exp(-\theta x)}{\alpha \theta^\alpha + \Gamma(\alpha)} \right] = 0$$
These results confirm that the proposed distribution has a mode.

3.2. Reliability Analysis
The survival function (or the reliability function) is the probability that a subject survives longer than the expected time. The survival function of NTPLD is given by

\[ S(x; \alpha, \theta) = 1 - F(x; \alpha, \theta) = \frac{\alpha \theta^\alpha e^{-\theta x} - \gamma(\alpha + 1, \theta x) + \Gamma(\alpha + 1)}{\alpha \theta^\alpha + \Gamma(\alpha + 1)} \]

The hazard function (also known as the hazard rate, instantaneous failure rate or force of mortality) is the probability to measure the instant death rate of a subject. Suppose \( X \) be a continuous random variable with pdf \( f(x) \) and cdf \( F(x) \). The hazard rate function of \( X \) is defined as

\[ h(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x / X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)} \]

The corresponding \( h(x) \) of NTPLD can be obtained as

\[ h(x; \alpha, \theta) = \frac{\theta^{\alpha+1}(\alpha + x^\alpha)e^{-\theta x}}{\alpha \theta^\alpha e^{-\theta x} - \gamma(\alpha + 1, \theta x) + \Gamma(\alpha + 1)} \]

The natures of survival function and the hazard function of NTPLD for varying values of parameters are shown graphically in figures 3 and 4, respectively.
3.3. Mean Residual Life Function

The mean residual life function of the NTPLD can be obtained as

\[ m(x; \alpha, \theta) = E[X - x \mid X > x] = \frac{1}{1 - F(x; \alpha, \theta)} \int_x^\infty \left[ 1 - F(t; \alpha, \theta) \right] dt \]

\[ = \frac{\alpha \theta^{\alpha - 1} \gamma(\alpha, \theta x) + \gamma(\alpha + 1, \theta x)}{\alpha \theta^{\alpha} e^{-\theta x} - \gamma(\alpha + 1, \theta x) + \Gamma(\alpha + 1)} - x \]

3.4. Moments and Related Measures

The \( r \)th moment about origin of NTPLD has been obtained as

\[ \mu_r' = \frac{\alpha \theta^{\alpha} \Gamma(\alpha + 1) + \Gamma(r + \alpha + 1)}{\theta^r [\alpha \theta^{\alpha} + \Gamma(\alpha + 1)]} ; \ r = 1, 2, ... \]  

(3.4.1)

Taking \( r = 1, 2, 3 \) and 4 in (3.4.1), the first four moments about origin are obtained as

\[ \mu_1' = \frac{\alpha \theta^{\alpha} \Gamma(\alpha + 1) + \Gamma(\alpha + 2)}{\theta [\alpha \theta^{\alpha} + \Gamma(\alpha + 1)]} \]

\[ \mu_2' = \frac{\alpha \theta^{\alpha} \Gamma(\alpha + 1) + \Gamma(\alpha + 3)}{\theta^2 [\alpha \theta^{\alpha} + \Gamma(\alpha + 1)]} \]

\[ \mu_3' = \frac{\alpha \theta^{\alpha} \Gamma(\alpha + 1) + \Gamma(\alpha + 4)}{\theta^3 [\alpha \theta^{\alpha} + \Gamma(\alpha + 1)]} \]

\[ \mu_4' = \frac{\alpha \theta^{\alpha} \Gamma(\alpha + 1) + \Gamma(\alpha + 5)}{\theta^4 [\alpha \theta^{\alpha} + \Gamma(\alpha + 1)]} \]

The first two central moments of NTPLD have been obtained as

\[ \mu_1 = 0 \]

\[ \mu_2 = \frac{\alpha^2 \theta^{2\alpha} + \alpha \theta^{\alpha} \Gamma(\alpha + 1)(\alpha^2 + \alpha + 2) + (\alpha + 1) \Gamma^2(\alpha + 1)}{\theta^2 [\alpha \theta^{\alpha} + \Gamma(\alpha + 1)]^2} \]
The standard deviation and the coefficient of variation and coefficient of dispersion of NTPLD are obtained as

\[
\text{Standard deviation } (\sigma) = \sqrt{\frac{\alpha^2 \theta^{2\alpha} + \alpha \theta^\alpha \Gamma(\alpha + 1)(\alpha^2 + \alpha + 2) + (\alpha + 1)\Gamma^2(\alpha + 1)}{\theta[\alpha \theta^\alpha + \Gamma(\alpha + 1)]}}
\]

\[
\text{Co-efficient of variation } \left( \frac{\sigma}{\mu_1} \right) = \frac{\sqrt{\alpha^2 \theta^{2\alpha} + \alpha \theta^\alpha \Gamma(\alpha + 1)(\alpha^2 + \alpha + 2) + (\alpha + 1)\Gamma^2(\alpha + 1)}}{\alpha \theta^\alpha \Gamma(\alpha + 1) + \Gamma(\alpha + 2)}
\]

\[
\text{Co-efficient of dispersion } \left( \frac{\sigma^2}{\mu_1^2} \right) = \frac{\alpha^2 \theta^{2\alpha} + \alpha \theta^\alpha \Gamma(\alpha + 1)(\alpha^2 + \alpha + 2) + (\alpha + 1)\Gamma^2(\alpha + 1)}{\theta[\alpha \theta^\alpha + \Gamma(\alpha + 1)][\alpha \theta^\alpha \Gamma(\alpha + 1) + \Gamma(\alpha + 2)]}
\]

The forms of the rest central moments and related measure like kurtosis and skewness are very big and complicated and thus are not given here. However, if required, can be obtained in the same way.

**4. Generating Functions**

Let \( X \) have a NTPLD, then the MGF of \( X \) is obtained as

\[
M_X(t) = E(e^{itX}) = \int_0^\infty e^{itx} f(x) \, dx
\]

Using Taylor’s series, we have

\[
M_X(t) = E(e^{itx}) = \int_0^\infty \left( 1 + tx + \frac{(tx)^2}{2!} + \cdots \right) f(x) \, dx
\]

\[
= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j g(x) \, dx = \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j
\]

The moment generating function of NTPLD is given by

\[
M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!} \frac{\alpha \theta^\alpha \Gamma(\alpha + 1) + \Gamma(\alpha + 1)}{\theta^j \left[ \alpha \theta^\alpha + \Gamma(\alpha + 1) \right]}
\]

Similarly, the characteristic function of NTPLD can be obtained as

\[
\Phi_X(t) = M_X(it) = \sum_{j=0}^\infty \frac{(it)^j}{j!} \frac{\alpha \theta^\alpha \Gamma(\alpha + 1) + \Gamma(\alpha + 1)}{\theta^j \left[ \alpha \theta^\alpha + \Gamma(\alpha + 1) \right]}
\]

The cumulant generating function is given by

\[
K_X(t) = \ln[M_X(it)] = \ln \left[ \sum_{j=0}^\infty \frac{(it)^j}{j!} \frac{\alpha \theta^\alpha \Gamma(\alpha + 1) + \Gamma(\alpha + 1)}{\theta^j \left[ \alpha \theta^\alpha + \Gamma(\alpha + 1) \right]} \right]
\]
5. Distribution of Order Statistics

Let \( x_1, x_2, \ldots, x_n \) be the random samples from NTPLD \((\alpha, \theta)\). The pdf of \( i^{th} \) order statistics is given by

\[
f_{i,n}(x) = \frac{n!}{(i-1)!(n-i)!} f_x(x) \left[ F_x(x) \right]^{-1} \left[ 1 - F_x(x) \right]^{-i}
\]

The pdf of \( i^{th} \) order statistics \( X_{(i)} \) of NTPLD is given by

\[
f_{i,n}(x) = \frac{n!}{(i-1)!(n-i)!} \theta^{\alpha+1} (\alpha + x^n) \exp(-\theta x) \left[ \frac{\alpha \theta^\alpha (1 - e^{-\theta x}) + \gamma (\alpha + 1, \theta x)}{\alpha \theta^\alpha + \Gamma(\alpha + 1)} \right]^{n-i} \frac{1}{\alpha \theta^\alpha + \Gamma(\alpha + 1)}
\]

The pdf of the first order statistic \( X_{(1)} \) can be expressed as

\[
f_{1,n}(x) = n \theta^{\alpha+1} (\alpha + x^n) \exp(-\theta x) \left[ \frac{\alpha \theta^\alpha (1 - e^{-\theta x}) + \gamma (\alpha + 1, \theta x)}{\alpha \theta^\alpha + \Gamma(\alpha + 1)} \right]^{n-1}
\]

The pdf of the largest order statistic \( X_{(n)} \) can be expressed as

\[
f_{n,n}(x) = n \theta^{\alpha+1} (\alpha + x^n) \exp(-\theta x) \left[ \frac{\alpha \theta^\alpha (1 - e^{-\theta x}) + \gamma (\alpha + 1, \theta x)}{\alpha \theta^\alpha + \Gamma(\alpha + 1)} \right]^{n-1}
\]

6. Maximum Likelihood Estimation

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from a NTPLD \((\alpha, \theta)\). The log-likelihood function can be expressed as

\[
\log L = n \left[ (\alpha + 1) \ln \theta - \ln \left( \alpha \theta^\alpha + \Gamma(\alpha + 1) \right) \right] + \sum_{i=1}^{n} \ln (\alpha + x_i^\alpha) - n \theta \bar{x}
\]

The maximum likelihood estimates (MLE) \((\hat{\alpha}, \hat{\theta})\) of parameters \((\alpha, \theta)\) of NTPLD are the solutions of the following log-likelihood equations

\[
\frac{\partial}{\partial \alpha} \log L = \sum_{i=1}^{n} \left[ \ln \theta - \frac{1}{\alpha \theta^\alpha + \Gamma(\alpha + 1)} \right] \left[ \alpha^2 \theta^{\alpha-1} + \theta^\alpha + \Psi(\alpha + 1) \right] + \frac{\alpha}{\sum_{i=1}^{n} x_i^\alpha} - \frac{n \theta}{\sum_{i=1}^{n} x_i^\alpha} - n \bar{x}
\]

\[
\frac{\partial}{\partial \theta} \log L = \frac{n(\alpha + 1)}{\theta} - \frac{n \alpha^2 \theta^{\alpha-1}}{\alpha \theta^\alpha + \Gamma(\alpha + 1)} - n \bar{x}
\]
where \( \Psi(\alpha + 1) = \frac{\partial}{\partial \alpha} \Gamma(\alpha + 1) \).

These log-likelihood equation can not be solved analytically and required statistical software with iterative numerical techniques. These equations can be solved using R-software.

The 2×2 observed information matrix of NTPLD can be presented as,

\[
\begin{pmatrix}
\frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\
\frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta^2}
\end{pmatrix}
\]

The inverse of the information matrix results in the well-known variance-covariance matrix. The 2×2 approximate Fisher information matrix corresponding to the above observed information matrix is given by

\[
I^{-1} = -E \begin{pmatrix}
\frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\
\frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta^2}
\end{pmatrix}
\]

The solution of the Fisher information matrix will yield asymptotic variance and covariance of the ML estimators for \((\hat{\alpha}, \hat{\theta})\). The approximate 100(1-\(\alpha\))% confidence intervals for \((\alpha, \theta)\) respectively are

\[
\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \frac{\sigma_{\alpha\alpha}}{n} \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\alpha}{2}} \frac{\sigma_{\theta\theta}}{n},
\]

where \(Z_{\frac{\alpha}{2}}\) is the upper 100\(\alpha\)th percentile of the standard normal distribution.

7. Applications

The new two parameter lindley distribution (NTPLD) has been fitted to a lifetime data-set. In this section, we present the fit of NTPLD and compare its goodness of fit with Weibull distribution (WD), two parameter Lindley-1 (TPLD-1), two parameter Lindley-2 (TPLD-2), Quasi-Lindley distribution (QLD), new Quasi-Lindley distribution (NQLD), Lindley distribution (LD) and exponential distribution (ED).

The following dataset has been considered.

The data set, strength data, which were originally reported by Badar and Priest [1] and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10 mm with sample size \((n = 63)\). This data set consists of observations:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937,
In order to compare NTLD with other distributions, we consider the criteria like Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC) and \( -2 \log L \). The better distribution corresponds to lesser values of AIC, BIC, AICC and \( -2 \log L \). The formulae for calculating AIC, BIC and AICC are as follows:

\[
AIC = 2k - 2\log L, \quad BIC = k \log n - 2\log L, \quad AICC = AIC + \frac{2k(k + 1)}{(n - k - 1)},
\]

where \( k \) is the number of parameters, \( n \) is the sample size and \( -2 \log L \) is the maximized value of log likelihood function. The ML estimates of the parameters of the considered distributions along with values of \( -2 \log L, AIC, AICC \) and \( BIC \) for the datasets are presented in Table 1.

8. Conclusions

A new two-parameter Lindley distribution (NTPLD) has been proposed. The nature of pdf, cdf, survival function, hazard rate function has been studied with varying values of parameters. Descriptive measures based on moments of NTPLD has been derived. The method of maximum likelihood has been discussed for estimating parameters. Fisher’s information matrix and confidence intervals of the parameters of the proposed distribution have been presented. The goodness of fit of NTPLD has been discussed with a real

<table>
<thead>
<tr>
<th>Distributions</th>
<th>ML parameters</th>
<th>(-2 \log L)</th>
<th>(AIC)</th>
<th>(AICC)</th>
<th>(BIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha) (\theta)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NTPLD</td>
<td>32.35558</td>
<td>10.50217</td>
<td>123.9867</td>
<td>127.9867</td>
<td>128.1867</td>
</tr>
<tr>
<td>WD</td>
<td>0.100000</td>
<td>2.244289</td>
<td>179.8587</td>
<td>183.8587</td>
<td>184.0587</td>
</tr>
<tr>
<td>TPLD-1</td>
<td>2.295784</td>
<td>6.536456</td>
<td>220.7134</td>
<td>224.7134</td>
<td>224.9134</td>
</tr>
<tr>
<td>TPLD-2</td>
<td>0.100000</td>
<td>0.634249</td>
<td>224.3504</td>
<td>228.3504</td>
<td>228.5504</td>
</tr>
<tr>
<td>QLD</td>
<td>0.100000</td>
<td>0.637224</td>
<td>226.2462</td>
<td>230.2462</td>
<td>230.4462</td>
</tr>
<tr>
<td>NQLD</td>
<td>2179.448</td>
<td>0.653649</td>
<td>220.7080</td>
<td>224.7080</td>
<td>224.9080</td>
</tr>
<tr>
<td>LD</td>
<td>0.539233</td>
<td>(-)</td>
<td>242.7152</td>
<td>244.7153</td>
<td>245.1220</td>
</tr>
<tr>
<td>ED</td>
<td>0.326873</td>
<td>(-)</td>
<td>266.8916</td>
<td>268.8915</td>
<td>269.2988</td>
</tr>
</tbody>
</table>

It is obvious from the goodness of fit of the two–parameter and one parameter distributions in table 1 that NTPLD gives much closer fit than other two parameter distributions.
lifetime data and compared with other two –parameter Lindley distribution and one parameter Lindley distribution and exponential distribution.

Acknowledgements
Authors are grateful for the fruitful comments from the three independent reviewers which improved the quality and presentation of the paper.

References:
3n+1 Problem and its Dynamics

1Bishnu Hari Subedi and 2Ajaya Singh
1, 2 Central Department of Mathematics, Institute of Science and Technology
Tribhuvan University, Kirtipur, Kathmandu, Nepal

Emails: subedi.abs@gmail.com, subedi_bh@cdmathtu.edu.np, bishnu.subedi@cdmathtu.edu.np, singh.ajaya1@gmail.com

Corresponding Author: Bishnu Hari Subedi

Abstract: The subject of this paper is the well-known 3n+1 problem of elementary number theory. This problem concerns with the behaviour of the iteration of a function which takes odd integers n to 3n + 1, and even integers n to \( \frac{n}{2} \). There is a famous Collatz conjecture associated to this problem which asserts that, starting from any positive integer n, repeated iteration of the function eventually produces the value. We briefly discuss some basic facts and results of 3n + 1 problem and Collatz conjecture. Basically, we more concentrate on the generalization of this problem and conjecture to holomorphic dynamics.

Keywords: 3n +1 problem, Collatz conjecture, holomorphic dynamics, Fatou set, Julia set, (pre)-periodic Fatou component, wandering domain.

AMS (2010) Subject Classification: 11Y55, 11B37, 30D05

1. Introduction

There are certain problems in mathematics which looks very simple and understandable to wide range of peoples, but very difficult to solve, and no one has yet found the solution to them. These problems are quite interesting because it seems the prerequisites for understanding the statement of the problem are much lower than the prerequisites for working on the problem. One of such problems is the 3n+1 problem. This is a well-known problem in elementary number theory, and it can be explained to a child who has learned how to divide by 2 and multiply by 3. The problem can be stated simply as follows: Take any positive integer n. If n is even, divide it by 2 to get \( \frac{n}{2} \). If n is odd, multiply it by 3 and add 1 to get 3n+1. Repeat the process again and again. The conjecture associated to this problem is called Collatz conjecture or simply 3n +1 conjecture, and it asserts that, no matter what the number n is taken, the process will always eventually reach 1, that is, every positive integer is eventually periodic, and the cycle it falls onto is 1→4→2→1. This conjecture is first proposed by Lothar Collatz in 1937. It is also known, the Ulam conjecture, the Kakutani’s problem, the Twaines conjecture, or the Syracuse problem. The problem looks very simple, but “mathematics has not been ready for such problems”, according to Paul Erdos [6]. He also offered US$500 for its solution [7]. Jeffrey Lagarias in 2010 claimed that, based only on known information about this problem, this is an extraordinarily difficult problem, completely out of research of present mathematics [10]. Therefore, the Collatz conjecture remains today unsolved problem of mathematics, as it has been for over 80 years. The interest in this problem extends past the area of Number Theory; including Computer Science, via algorithms to help compute and find patterns in our iteration, into Logic as decision problems, and Dynamical Systems, by examining our iteration as a dynamical system on set of integer’s \( \mathbb{Z} \). A systematic survey with variety of already established results about 3n +1 problem can be found in the work of Lagarias [9], [10], and Wirsching [21]. The classical Collatz conjecture has been extensively studied by several researchers [5], [16].
In this paper, we more concentrate on the fact of extending this problem to holomorphic dynamics. The related research had been done first by Letherman et al. [13] in 1999 and a little bit more extension by Lakshminarayanan and Ramohan [11] in 2012.

2. Mathematical Formulation of the 3n + 1 Problem

We first define some useful functions and concepts to describe behaviour of the sequences and starting values in the 3n + 1 problem.

Definition 2.1 (3n + 1 function): Let \( \mathbb{N} \) represents the set of natural numbers. For any \( n \in \mathbb{N} \), the 3n + 1 or Collatz function \( f: \mathbb{N} \rightarrow \mathbb{N} \) is defined by \( f(n) = 3n + 1 \) if \( n \) is odd and \( f(n) = \frac{n}{2} \) if \( n \) is even. In modular notation, function \( f \) can be written as \( f(n) = 3n + 1 \) if \( n \equiv 1 \pmod{2} \), and \( f(n) = \frac{n}{2} \) if \( n \equiv 0 \pmod{2} \).

There are some terminologies which are defined by using 3n + 1 function.

The trajectory or orbit \( O^r(n) \) of \( n \) is the ordered set \( \{n, f(n), f^2(n), f^3(n), \ldots\} \), where \( f^i \) represents \( i \)th composition of \( f \) with itself, and it is usually known as \( i \)th iterates of \( f \). If \( |O^r(n)| = \infty \), then \( O^r(n) \) is said to be a divergent trajectory. If \( |O^r(n)| = k < \infty \) and \( f^k(n) = n \), then \( O^r(n) \) is said to be a cycle of length \( k \).

The number of steps needed to iterate below \( n \): \( \gamma(n) = \inf \{k : f^k(n) < n\} \) is called the stopping time of \( n \). The number of steps needed to iterate 1: \( \sigma(n) = \inf \{k : f^k(n) = 1\} \) is called the total stopping time. The largest number to which \( n \) iterates: \( h(n) = \sup \{f^k(n) : k \in \mathbb{N}\} \) is called the height of \( n \).

With these definitions, the 3n + 1 or Collatz conjecture is formulated as follows.

Conjecture 2.1 (3n + 1 or Collatz conjecture): For every \( n \in \mathbb{N} \), there exists a \( k \in \mathbb{N} \) with \( f^k(n) = 1 \).

Conjecture 2.1 asserts that every \( n \) has a finite total stopping time. If, for some \( n \), such a \( k \) doesn't exist, we say that \( n \) has infinite total stopping time and the conjecture is false. If the conjecture is false, it can only be because of there is some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound. No such sequence has been found.

Let \( a_0 = f^k(n) \), then \( a_0 = n, a_1 = f(n) = f(a_0), a_2 = f(a_1) = f^2(n) \), and so on, with \( a_0 \geq 1 \). According to Conjecture 1.1, any \( n \geq 1 \) would always eventually arrive at \( a_i = 1 \) for some \( i = 1, 2, \ldots \) after which it will stay in the cycle \( (1, 4, 2, 1) \) forever. For example, if we look at the starting values \( a_0 = 1 \) to \( 9 \), we get the following sequences:

\[
\begin{align*}
a_0 &= 1; \{a_0, a_1, a_2, \ldots\} = \{1, 4, 2, 1, \ldots\}. \\
a_0 &= 2; \{a_0, a_1, a_2, \ldots\} = \{2, 1, \ldots\}. \\
a_0 &= 3; \{a_0, a_1, a_2, \ldots\} = \{3, 10, 5, 16, 8, 4, 2, 1, \ldots\}. \\
a_0 &= 4; \{a_0, a_1, a_2, \ldots\} = \{4, 2, 1, \ldots\}. \\
a_0 &= 5; \{a_0, a_1, a_2, \ldots\} = \{5, 16, 8, 4, 2, 1, \ldots\}. \\
a_0 &= 6; \{a_0, a_1, a_2, \ldots\} = \{6, 10, 5, 16, 8, 4, 2, 1, \ldots\}. \\
a_0 &= 7; \{a_0, a_1, a_2, \ldots\} = \{7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \ldots\}. \\
a_0 &= 8; \{a_0, a_1, a_2, \ldots\} = \{8, 4, 2, 1, \ldots\}. \\
a_0 &= 9; \{a_0, a_1, a_2, \ldots\} = \{9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \ldots\}. 
\end{align*}
\]

However, when we look at \( a_0 = 27 \), we get the following sequence: \( \{a_0, a_1, a_2, \ldots\} = \{27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 414, 206, 103, 310, 155, 466, 233, \ldots\} \).
700, 350, 175, 526, 263, …}. Surprisingly, it takes 111 iterations to reach 1, and the largest number that we get in 77th iterations is 9232. That is, \( \sigma (27) = 111 \), \( h (27) = 9232 \), and \( \gamma (27) = 96 \). As we see in the above sequence, the number 9 has longest total stopping time 19 for all \( n \in \mathbb{N} \) below 10. Likewise, below 100, number 97 has longest total stopping time 118; below 1000, number 871 has longest total stopping time 178; below 10000, number 6171 has longest total stopping time 261. Below \( 10^{10} \), number 9780657630 has longest total stopping time 1132[12], and for the numberless than \( 10^{17} \), 93571393692802302 has longest total stopping time 2091[17]. Until 2020, the conjecture has been checked by computer for all starting values upto \( 2^{68} \approx 2.95 \times 10^{20} \). Please note that the computer evidence is not a proof that the conjecture is true. As shown in the cases of the Pólya conjecture [15], the Mertens conjecture, and Skewes' number ([18], [19]), counterexamples were found when using very large numbers.

Mathematically speaking, a (discrete) dynamical system is a state space \( X \), together with a shift map \( f \) from \( X \) to itself. The iterates \( f, f^2, f^3, \ldots \) describe the dynamics of the system. In the \( 3n + 1 \) dynamical system, the state space is the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots \} \) and the shift map is the \( 3n + 1 \) map \( f \). The \( 3n + 1 \) conjecture 2.1 highlights the basic fact that even very simple equations can lead to amazingly complicated dynamics. In this paper, we behave \( 3n + 1 \) problem as a discrete integer dynamics, and we see the latest results that are extended up to holomorphic dynamics. Lagarias [9] extended this problem to the set of rational numbers \( \mathbb{Q} \) by defining the maps:

\[
f_k (x) = \left\{ \begin{array}{ll}
\frac{3x+k}{2} & \text{if } x \equiv 1 \pmod{2}, \\
\frac{x}{2} & \text{if } x \equiv 0 \pmod{2},
\end{array} \right.
\]

for any entire holomorphic function \( h \). Note that first two terms of (2.3) match with (2.2), and this function agrees on all integers with the \( 3n + 1 \) function. The map (2.3) is thus known as the holomorphic \( 3n + 1 \) function. Conjecture 2.1 was reformulated by them in this context as follows.

**Conjecture 2.2** (The holomorphic \( 3n + 1 \) conjecture): Iterating the function \( f \) of (2.3) on any positive integer will land at the number 1 after finitely many steps.

Tempkin [20] extended the \( 3n + 1 \) problem to the set of real numbers \( \mathbb{R} \) by defining the function:

\[
f(x) = \left\{ \begin{array}{ll}
-(5n - 2)x + n(10n - 3), & \text{if } x \in [2n - 1, 2n], \ n \in \mathbb{N} \\
(5n + 4)x - n(10n + 7), & \text{if } x \in [2n, 2n + 1], \ n \in \mathbb{N}
\end{array} \right.
\]

He proved that on each interval \([n, n+1], n \in \mathbb{N}\), \( f \) has periodic points of every possible period. Chamberland studied similar extension to (2.1) by defining the function:

\[
f(x) = \frac{x}{2} \cos \frac{\pi x}{2} + 3x \cos \frac{\pi x}{2} = x + \frac{1}{4} \frac{(2x + 1)}{4} \cos(\pi x)
\]

He showed that any cycle on \( \mathbb{R} \) must be locally attractive. This map has two attracting cycles, namely \((1, 2)\) and \((1.192531907\ldots, 2.138656335\ldots)\). Equivalent to \( 3n + 1 \) problem, he conjecture that these are only two attracting cycle of \( f \) on \( \mathbb{R}^+ \).

Letherman et al. [13] extended the map (2.2) to the set of complex numbers \( \mathbb{C} \) by defining the function:

\[
f(z) = \frac{z}{2} + \frac{1}{2} (1 - \cos \pi z) \left(z + \frac{1}{2}\right) + \frac{1}{\pi} \left(1 - \cos \pi z\right) \sin \pi z + h(z) \sin^2 \pi z
\]

for any entire holomorphic function \( h \).
3. Dynamics of a Holomorphic Function

In this section, we briefly review the notion of holomorphic dynamics. For more details, we refer [4], [8], [14]. The main purpose of this section is to provide a general background for the dynamics of the holomorphic 3n + 1 function (2.3) of Section 2. Note that our holomorphic 3n + 1 function (2.3) is transcendental entire, so we mainly concern with results related to transcendental dynamics.

Let \( f \) be a holomorphic function. We can define orbit and cycle of \( f \) as defined in Section 2. Note that \( f^n(z) \) is well defined for all \( z \) except for a countable set which consists of the pole of \( f, f^2, f^3, \ldots, f^{n-1} \) when \( f \) is meromorphic. The study of the iterative sequence \((f^n(z))\) for various initial state \( z \) is known as holomorphic dynamics. The major concern of holomorphic dynamics is to study the fate of these orbits in the sense that fate is predictable or not. That is, the goal of studying holomorphic dynamics is to describe the asymptotic or long term behaviour of the sequence \((f^n(z))\). The dynamics of a holomorphic map mostly concerns with a dichotomy of the complex plane \( \mathbb{C} \) into disjoint subsets where the sequence \((f^n(z))\) as \( n \to \infty \) shows a normal or chaotic behaviour.

A subset of \( \mathbb{C} \) where \((f^n(z))\) as \( n \to \infty \) behaves normally (in the sense of Montel) is known as Fatou set \( F(f) \), and its complement is called Julia set \( J(f) \). A maximum domain of normality of the iterates of \( f \), that is, a connected component of the Fatou set is known as a stable domain or Fatou component. Fatou set is open by definition, and so its complement Julia set is closed. A Fatou component is simply or multiply connected. The following assertions are about the simply connected Fatou components.

**Theorem 3.1:** Let \( f \) be a transcendental entire function bounded on a curve \( \Gamma \) which tends to \( \infty \). Then every component of \( F(f) \) is simply connected.

**Theorem 3.2:** Every unbounded Fatou component of a transcendental entire function is simply connected.

**Theorem 3.3:** Let \( f \) be a transcendental entire function such that \( F(f) \) contains an unbounded component \( D \). Then every components of \( F(f) \) is simply connected.

The dynamics of a holomorphic function, in large extent, is determined by the periodicity of a point. A point \( z \) is called periodic if \( f^n(z) = z \) for some positive integer \( n \). The smallest \( n \) is called its period. In particular, if \( f(z) = z \), then \( z \) is called fixed point (or equilibrium position) of \( f \). The point \( z \) is called pre-periodic (or eventually periodic) if \( f^{k+n}(z) = f^k(z) \) for some \( k, n > 0 \) and strictly pre-periodic if it is pre-periodic but not periodic. Let \( z \) is a periodic point of period \( n \) with \((f^n)(z) = \lambda \), where the prime (‘) denotes the complex differentiation. The complex number \( \lambda \) obtained in this way is called multiplier or eigenvalue. The point \( z \) is called attractive or stable (or super-attractive) if \( |\lambda| < 1 \) (or \( |\lambda| = 0 \)), and in this case, nearby points are attracted to the orbit under iteration by \( f \), repelling or unstable if \( |\lambda| > 1 \) in which case, points close to the orbit move away, and indifferent (or neutral) if \( |\lambda| = 1 = e^{2\pi i \theta} \) in which case, iteration of nearby points stay near \( z \) but not converge to \( z \). When \( \theta \) is rational number (in this case \( \lambda^n = 1 \) for some integer \( n \)), the periodic point is called parabolic (or rationally indifferent), and the nearby dynamics are completely known. When \( \theta \) is irrational (in this case \( \lambda^n \neq 1 \)), the periodic point \( z \) is called irrationally indifferent, and there are certain values of \( \theta \), where nearby dynamics is still not known. It is known that a non-constant and non-linear entire function has at least two periodic points of period 1 or 2. This fact is generalized to the following assertion.

**Theorem 3.4[3]:** A (transcendental) entire function has infinitely many (repelling) periodic points of period \( n \) (or \( f^n \) has infinitely many fixed points) for all \( n \geq 2 \).
This assertion was first proved by Rosenbloom in 1948, and later it was corrected by adding ‘repelling’ before periodic by Bergweiler in 1991. From Theorem 3.1, one can say that transcendental entire function need not have fixed points. For example, \( f(z) = e^z + z \) has no fixed points. Also, transcendental entire function need have attracting periodic points. For example, \( f(z) = e^z \) has no attracting periodic points.

The periodic points that we categorized above are contained in Fatou or Julia sets as shown in the following assertion.

**Theorem 3.5**: Let \( f \) be a holomorphic function. Then \( F(f) \) contains all (super) attracting periodic points and cycles, \( J(f) \) contains all repelling periodic points and cycles, all rationally indifference periodic points and cycles.

There are two types of points for which the inverse of holomorphic functions are not well defined, namely critical values and asymptotic values, and collectively they are known as singular values.

**Definition 3.1 (Critical value, asymptotic value and singular value)**: Let \( f \) be a holomorphic function. The critical point of a function \( f \) is a point \( z_0 \) such that \( f'(z_0) = 0 \) and the critical value is the image of critical point under \( f \). A point \( w \in \mathbb{C} \) is said to be an asymptotic value if there is a curve \( \gamma \) tending to \( \infty \) such that, along \( \gamma \) the values of \( f(z) \) converges to \( w \). The closure of the set of critical and asymptotic values is known as the set of singular values. This set is usually denoted by \( \text{SV}(f) \).

Note that among the entire functions, only transcendental entire functions may have asymptotic values. It is clear that polynomials cannot have finite asymptotic values. There are certain holomorphic functions whose finite asymptotic values can also be critical values. For example, \( f(z) = z^2 e^{-z^2} \) has an asymptotic value 0, and critical values 0, 1/e.

**Definition 3.2 (Finite type and bounded type holomorphic function)**: Let \( f \) be a holomorphic function. If \( \text{SV}(f) \) is finite, then \( f \) is said to be finite type. If \( \text{SV}(f) \) is bounded, then \( f \) is said to be bounded type.

Note that any finite type function is necessarily bounded type, but the converse may not hold. For example, \( f(z) = z^2 e^{-z^2} \) is finite type transcendental entire function, and hence it is also bounded type. Later, in Section 4, we will discuss that the holomorphic \( 3n + 1 \) function (2.3) is bounded type but not finite type.

There are dynamically important different Fatou components. For transcendental entire functions, a Fatou component \( U \) is one of the following type, for more details, we refer [4], [8], [14].

- **Periodic component**: if \( f^n(U) \subseteq U \). Minimum \( n \) is called the period of \( U \).
- **Pre-periodic component**: if \( f^n(U) \) is periodic for some integer \( n \).
- **Wandering domain**: if \( \{f^n(U)\} \) are disjoint for all \( n \).

Periodic component \( U \) is one of the following types.

- **Immediate attracting basin**: if \( U \) contains (supper) attracting periodic point.
- **Parabolic or Leau domain**: if \( \partial U \) contains periodic point.
- **Siegel disk**: if \( U \) contains irrationally indifference periodic point.
- **Baker domain**: if \( U \) is unbounded and dynamics converge to \( \infty \) locally uniformly.

There are certain Fatou components which are always simply connected as shown in the following assertions.
Theorem 3.6[1]: Let \( f \) be a transcendental entire map. Then every periodic or pre-periodic Fatou component is simply connected, and therefore any multiply connected Fatou component is bounded and wandering.

Theorem 3.7[2]: Let \( f \) be a bounded type transcendental entire function. Then all component of \( F(f) \) are simply connected.

For certain holomorphic functions, Baker and wandering domains do not exists as shown in the following assertion.

Theorem 3.8[2]: Finite type transcendental entire functions do not have wandering domains, and bounded type transcendental entire function do not have Baker domain.

For certain holomorphic function, Fatou set can be empty as shown in the following assertions.

Theorem 3.9: Let \( f \) be a finite type holomorphic function such that orbit of each point in \( SV(f) \) either is pre-periodic or converge to \( \infty \). Then \( F(f) = \emptyset \). In particular, if \( f(z) = e^z \), then \( F(f) = \emptyset \).

Definition 3.3: (Forward, backward and completely invariant set): Let \( f \) be a function. A set \( E \) is said to be forward invariant if \( f(E) \subseteq E \), backward invariant if \( f^{-1}(E) \subseteq E \), and completely invariant if it is both forward and backward invariant.

For example, \( f(z) = z^2 \) has two completely invariant domains, namely \( \{z \in \mathbb{C} : |z| < 1\} \) and \( \{z \in \mathbb{C} : |z| > 1\} \) as Fatou components, and Julia set \( J(f) = \{z \in \mathbb{C} : |z| = 1\} \) which is also completely invariant. This is an example of entire function which has a bounded and an unbounded completely invariant Fatou component.

However, in case of transcendental entire functions, we have the following assertions.

Theorem 3.10: Let \( f \) be a transcendental entire function. Then a completely invariant Fatou component \( U \) is unbounded, and \( J(f) = \partial U \) and \( f \) has at most one completely invariant Fatou component.

For any holomorphic function, Fatou and Julia sets are themselves completely invariant as shown in the following assertion.

Theorem 3.11: Let \( f \) be a holomorphic map. Then Fatou and Julia sets are completely invariant.

4. Dynamics of the Holomorphic 3n+1 Function

In this section, we examine the dynamical behaviour of the holomorphic 3n+1 function (2.3) of Section 2.

We can easily check that \( z = 0 \) is a fixed point of the holomorphic 3n+1 function (2.3). If we differentiate holomorphic 3n+1 function (2.3), we get

\[
 f'(z) = \left[ \frac{\pi}{2} (z + \frac{1}{2}) + 2 \sin \pi z + 2 \pi h(z) \cos \pi z + h'(z) \sin \pi z \right] \sin \pi z
\]  

(4.1)

From (4.1), we can say that all integers are critical points of function (2.3). Also, \( |\lambda| = |f'(0)| = 0 \). Therefore, \( z = 0 \) is a super attracting fixed point, and thus it is in the Fatou set \( F(f) \) (by Theorem 3.5). If we consider function \( h(z) \) vanishes from function \( f(z) \), then function (4.1) reduces to

\[
 f'(z) = \left[ \frac{\pi}{2} (z + \frac{1}{2}) + 2 \sin \pi z \right] \sin \pi z.
\]  

(4.2)

For any \( z = n + \delta \) with \( |\delta| < \frac{1}{2\pi n} \), \( n \in \mathbb{Z} \), one can calculate \( |f'(z)| < \frac{1}{2} \).
By these facts, Letherman et al. ([13], Lemmas 3.1, 3.2) proved the following assertions.

**Theorem 4.1** ([13], Lemmas 3.1 and 3.2): Let \( f \) be a holomorphic \( 3n + 1 \) function (2.3). Then all integers are critical points. If entire function \( h \) in (2.3) vanishes everywhere, then all integers are in the Fatou Set.

From (4.1), we also can say that the holomorphic \( 3n + 1 \) function (2.3) is not finite type, but of course, bounded type. We can use this fact to prove the following assertions.

**Theorem 4.2** ([13], Proposition 3.4): Every Fatou component of the holomorphic \( 3n +1 \) function (2.3) is simply connected.

**Proof:** The holomorphic \( 3n +1 \) function (2.3) is bounded type. Hence, by Theorem 3.7, every component of \( F(f) \) is simply connected. \( \square \)

**Proposition 4.1:** Holomorphic \( 3n +1 \) function (2.3) has no Baker domains.

**Proof:** The function (2.3) is a bounded type transcendental entire function. By Theorem 3.8, it does not have Baker domain. \( \square \)

Theorem 4.2 was stated by Letherman et al. [13] but its above short proof is ours. Their proof of this theorem is different and little bit long. We stated and proved Proposition 4.1 ourselves. We can also justify the essence of Proposition 4.1 by the following assertion.

**Theorem 4.3** ([13], Proposition 3.6)): Let \( f \) be a holomorphic \( 3n +1 \) function (2.3). No domain at infinity can intersect the real line. In particular, no integer can be in a domain at infinity.

Since orbit of any \( n \in \mathbb{Z} \) under the holomorphic \( 3n + 1 \) function (2.3) contained in \( \mathbb{Z} \). And, since all inters are supper attracting because they are all critical points of \( f \). No orbit in a Siegel disk is discrete, so there is no chance of consisting integers in the Siegel disk as in the following assertion.

**Theorem 4.4** ([13], Lemma 3.3): Let \( f \) be a holomorphic \( 3n +1 \) function (2.3). If a Fatou component of \( f \) corresponding to an attracting orbit contains an integer, then this orbit is super attracting. No Fatou component corresponding to rational indifference orbit or to a Siegel disk can contain integers.

The holomorphic \( 3n +1 \) function (2.3) is not finite type, so it may or may not have wandering domains. By Theorem 4.1, all integers are in Fatou set, and by Theorem 3.5, (super) attracting periodic points and cycles are in Fatou set. Therefore, every integer is in the basin of attraction of a (super) attracting periodic orbit of integers, or in a wandering Fatou component. By Theorem 4.2, every Fatou component is simply connected, so there is no chance of existing multiply connected wandering domains. In this context, holomorphic \( 3n +1 \) function can have simply connected wandering domains only when Conjecture 2.2 is disproved. Letherman et al. [13]. Conjecture 3.7] conjectured that wandering domain for such a function does not exist.

**Conjecture 4.1:** The holomorphic \( 3n +1 \) function (2.3) has no simply connected wandering domain intersecting the integers.

If this conjecture was proved, then the holomorphic \( 3n + 1 \) Conjecture 2.1 is proved, and then the \( 3n + 1 \) Conjecture 2.1 is also proved. Finally, we can say that the fate of the famous Collatz conjecture depends upon the Conjecture 4.1.
References

[18] Skewes S. (1933), On the difference \( \pi (x) - \text{li}(x) \), *Journal of the London Mathematical Society*, **8**: 277–283.

□□
An Alternative Proof of Rubin's Lemma

Santosh Ghimire

Department of Applied Sciences and Chemical Engineering
Pulchowk Campus, Tribhuvan University, Kathmandu, Nepal
Email: santoshghimire@ioe.edu.np

Abstract: Rubin's Lemma is an inhomogenous type inequality which is satisfied by the sequence of dyadic martingales. In this paper, we give a proof using the measure theoretic approach which is simpler and different than the original probabilistic approach.

Keywords: Martingales, Law of the iterated logarithm, Quadratic characteristic, Dyadic intervals

1. Introduction

Rubin's Lemma is an inequality associated to a sequence of dyadic martingales and plays a vital role in the law of the iterated logarithm of dyadic martingales and harmonic functions. These law of the iterated logarithm give the asymptotic behaviour of the sequence of both functions: dyadic martingales and harmonic functions. We can study the role of Rubin's Lemma in the asymptotic behaviour of these functions in the various papers, for instances we refer a few [2] and [5]. In these papers, we find applications of the lemma in order to estimate the size of various sets in the law of the iterated logarithm. There are much more literature available related to the law of the iterated logarithm in various contexts, for instances [1], [3], [4], [6] and [8]. The proof of the Lemma can be found in [2] in which the proof uses the probabilistic approach. In the present work, we give a proof of the Lemma using the measure-theoretic approach. The original proof uses the probabilistic tools of statistics whereas our method uses the simple tools of measure theory which can be easily followed and thus the new proof is simpler in comparison with the original proof. Moreover, our method provides the proof of the lemma in the setting of mathematical analysis.

2. Definitions and Notations

Before proceeding with the main work, we first fix some notation and give some basic definitions which will be used in the course of proof.

A dyadic subinterval of the unit interval $[0,1]$ is an interval of the form $\left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right]$ where $n = 0,1,2,$ and $j = 0,1,...,2^n - 1$.

A dyadic martingale is a sequence of integrable functions $\{f_n\}_{n=0}^\infty$ where $f_n: [0,1) \to \mathbb{R}$ such that

a. for every $n$, $f_n$ is $\Gamma_n$ measurable where $\Gamma_n$ is the sigma algebra generated by the dyadic intervals of the form $\left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right]$ for $j = 0,1,...,2^n - 1$.

b. conditional expectation $E(f_{n+1} | \Gamma_n) = f_n$ where

$$E(f_{n+1} | \Gamma_n)(x) = \frac{1}{|Q|} \int_{Q_n} f_n(y) dy$$ with $|Q_n| = \frac{1}{2^n}, x \in Q_n.$
Associated with a sequence of dyadic martingales \( \{f_n\}_{n=0}^{\infty} \), we define:

i. **increments:** \( d_k = f_k - f_{k+1} \). So \( f_n(x) = \sum_{k=1}^{n} d_k(x) + f_0 \).

ii. **quadratic characteristics or square function:** \( S_n^2 f(x) = \sum_{k=1}^{n} d_k^2(x) \).

iii. **limit function:** \( S^2 f(x) = \lim_{n \to \infty} S_n^2 f(x) = \sum_{k=1}^{\infty} d_k^2(x) \).

We now state the inequality associated to the dyadic martingales and this inequality is called Rubin’s Lemma according to [7].

**Rubin’s Lemma:** For a dyadic martingales \( \{f_n\}_{n=0}^{\infty} \) with \( f_0 = 0 \), we have

\[
\int_{0}^{1} \exp \left( f_n(x) - \frac{1}{2} S_n^2 f(x) \right) dx \leq 1.
\]

We now give the proof the lemma using a different approach than the original proof.

3. **Proof of the Main Result**

Let us first define

\[
g(n) = \int_{0}^{1} \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) dx.
\]

We now claim that the function \( g(n) \) is a decreasing function of \( n \). Let \( Q_{nj} \) be an arbitrary \( n \)-th generation dyadic interval. One can see that \( \sum_{k=0}^{n} d_k(x) = f_n \) and \( f_n \) is constant on \( Q_{nj} \).

Using this fact, we have

\[
g(n + 1) = \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp \left( \sum_{k=0}^{n+1} d_k(x) - \frac{1}{2} \sum_{k=0}^{n+1} d_k^2(x) \right) dx.
\]

\[= \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx.
\]

\[= \sum_{j=0}^{2^n} \left[ \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \right]_{Q_{nj}} \int_{Q_{nj}} \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right)
\]

Let \( Q'_{(n+1)j} \) and \( Q''_{(n+1)j} \) be the dyadic subintervals of \( Q_{nj} \). Suppose \( d_{n+1} \) takes the value \( \alpha \) on \( Q'_{(n+1)j} \).

Then by expectation condition, \( d_{n+1} \) takes the value \( -\alpha \) on \( Q''_{(n+1)j} \).

This gives

\[\int_{Q_{nj}} \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right)
\]

\[= \int_{Q'_{(n+1)j}} \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx + \int_{Q''_{(n+1)j}} \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx
\]

52
\[= \int_{\theta'_{(n+1)}} \exp \left( \alpha - \frac{1}{2} \alpha^2 \right) dx + \int_{\theta''_{(n+1)}} \exp \left( -\alpha - \frac{1}{2} \alpha^2 \right) dx\]

\[= \left[ \exp \left( \alpha - \frac{1}{2} \alpha^2 \right) + \exp \left( -\alpha - \frac{1}{2} \alpha^2 \right) \right] \frac{1}{2^{n+1}}\]

\[= 2 \exp \left( -\alpha^2 \frac{1}{2} \right) \left( e^{\alpha} + e^{-\alpha} \right) \frac{1}{2^{n+1}}\]

Thus, \(\int_{Q_n} \exp \left( d_{n+1} (x) - \frac{1}{2} d_{n+1}^2 (x) \right) = 2 \exp \left( -\frac{\alpha^2}{2} \right) \cosh \alpha \frac{1}{2^{n+1}}\).

Now using the elementary fact that \(\cosh x \leq \exp \left( \frac{x^2}{2} \right)\), we have

\[g(n+1) \leq \sum_{j=0}^{2^n} \left[ \exp \left( \sum_{k=0}^{n} d_k (x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2 (x) \right) \right] \quad 2 \exp \left( -\frac{\alpha^2}{2} \right) \exp \left( \frac{\alpha^2}{2} \right) \frac{1}{2^{n+1}}\]

\[g(n+1) \leq \sum_{j=0}^{2^n} \left[ \exp \left( \sum_{k=0}^{n} d_k (x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2 (x) \right) \right] |Q_n|\]

\[g(n+1) \leq \sum_{j=0}^{2^n} \int_{Q_n} \exp \left( \sum_{k=0}^{n} d_k (x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2 (x) \right) dx\]

\[g(n+1) \leq g(n)\]

Let \(Q_{11}\) and \(Q_{12}\) be the dyadic subintervals of \(Q_0\). Assume that \(d_1\) takes value \(\theta\) on \(Q_{11}\) so that it takes value \(-\theta\) on \(Q_{12}\).

Now \(g(1) = \int_{0}^{1} \exp \left( d_1 (x) - \frac{1}{2} d_1^2 (x) \right) dx\)

\[= \int_{0}^{1} \exp \left( \theta - \frac{1}{2} \theta^2 \right) dx + \int_{1/2}^{1} \exp \left( -\theta - \frac{1}{2} \theta^2 \right) dx\]

\[= \exp \left( \theta - \frac{1}{2} \theta^2 \right) \frac{1}{2} + \exp \left( -\theta - \frac{1}{2} \theta^2 \right) \frac{1}{2}\]

\[= \exp \left( -\frac{1}{2} \theta^2 \right) \frac{e^\theta + e^{-\theta}}{2}\]

\[= \exp \left( -\frac{1}{2} \theta^2 \right) \cosh \theta\]

\[g(1) \leq \exp \left( -\frac{1}{2} \theta^2 \right) \exp \left( \frac{1}{2} \theta^2 \right) = 1.\]

Since \(g(n)\) is decreasing and \(g(1) \leq 1\), we conclude that \(g(n) \leq 1\).

This gives

\[\int_{0}^{1} \exp \left( \sum_{k=0}^{n} d_k (x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2 (x) \right) dx \leq 1.\]
Hence, \( \int_0^1 \exp \left( f_n(x) - \frac{1}{2} S_n^2 f(x) \right) dx \leq 1. \)
This proves our result.
Finally we note that if we rescale the function by some \( \lambda \), we have
\[
\int_0^1 \exp \left( \lambda f_n(x) - \frac{1}{2} \lambda^2 S_n^2 f(x) \right) dx \leq 1.
\]
This shows that the Rubin's Lemma is an inhomogeneous type inequality.

**Conclusion**

In this paper, we derived an inequality associated to a sequence of dyadic martingales which is popularly known as Rubin's Lemma. We used the measure theoretic approach to prove the inequality which is different than the original probabilistic approach.

**Acknowledgement**

The author would like to thank the referees for valuable insightful comments that helped to improve the paper.

**References**


Determinants of Households’ Adaptation Practices against Climate Change Impact on Off-farm Activities in Western Hill of Nepal

Ananta Raj Dhungana1*, Vikash Kumar KC2, Purna Bahadur Khand3 and Surya Mani Dhungana4
1School of Development and Social Engineering, Pokhara University, Pokhara, Nepal
2Department of Statistics, Tribhuvan University, PN Campus, Pokhara, Nepal
3School of Business, Pokhara University, Pokhara, Nepal
4Agriculture and Forestry University, Rampur, Nepal

Email: anantastat@gmail.com, vkkc2001@gmail.com, purnabahadurkhand@gmail.com,
smdhungana@afu.edu.np

Corresponding Author: Ananta Raj Dhungana

Abstract: Climate change is one of the serious concerns which have a substantial impact in all areas of human civilization. Among these areas, agriculture is the worst hit sector. This study aims to analyze the determinants of households’ adaptation practices against climate change impact on off-farm activities in western hill of Nepal. It utilizes the data collected from six village development committees from three districts (Lamjung, Tanahu and Kaski) of western hill of Nepal. Out of 245793 Households 556 households were chosen for the survey. A systematic random sampling technique was used to select the respondents. Data was collected using pre-tested structured questionnaire through face to face interview with household head or a household member having age 45 years and above with residing in that locality since last 15 years. Binary logistic regression analysis has been carried out. Household size is only the matter for determining the households' adaptation practices for shifting to non-agricultural activities which is also the matter for temporary migration to another places. Education is the determining factor for changing the food consumption habit and temporary migration. Agriculture skill is the common determining factor for change in food consumption habit and shifting to non-agricultural activities. Sex and marital status are the major determining factors only for shifting to non-agricultural employment. Age is the significant predictor of temporary migration. Caste is also the major determining factor for changing the food consumption habit and temporary migration respectively.

Keywords: Food consumption, Non-agriculture activities, Non-agriculture employment, Off-farm, Temporary migration.

1. Introduction

Climate change is a universal phenomenon affecting various sectors in the world and is reflected to be one of the most serious threats to human civilization. Globally, an exceptional increase in greenhouse emissions has led to increased climate change impacts. Agriculture is one the most influenced sectors due to the climate change. Farmers have been facing unfavorable production situation due to the impact of climate change all over the world and this is more intense where the agriculture related infrastructure are weak and not well developed (Lien, Kumbhakar & Hardaker, [8]). Such unfavorable situations increase the cost of production creating an inability to sell the production in competitive markets. So the farmers seek to adapt alternative options that have low risk of loss. Off farm activities may be the safe exit for avoiding the risks of climate change impact among farmers.
Nepal is one of the most vulnerable countries in terms of climate change, due to greater warming in recent years than that of the global trend. While there was 0.6°C global mean surface temperature rise, from 1975 to 2005, Nepal experienced a considerably higher temperature rise of 1.5°C (0.06°C per year) during a similar duration of time, from 1982 to 2006 (Shrestha, Gautam and Bawa, [12]). Similarly, the rainfall patterns are also becoming more erratic and decreasing (Wang, Yoon, Gillies and Cho, 2013). The mean rainfall has been decreasing by 3.7 mm (~3.2%) per month, per decade (MoE, [10]). These conditions have created a drought especially for the rain-fed hill farming system, where people depend on summer and winter rainfall for their major agricultural activities (Ghimire, Shivakoti & Perret, [5]). Moreover, the mean annual temperature is predicted to be increased between 1.3 °C to 3.8 °C by the 2060s, and 1.8 °C to 5.8 °C by the 2090s while annual precipitation reduction could be within the range of 10% to 20%, across the country (MoE, [10]).

Adaptation is an adjustment in natural or human systems in response to actual or expected climatic conditions or risks and can be regarded as a policy option to contain the negative effects of climate change (Kurukulasuriya and Mendelsohn, [7]). Adaptation is the most efficient way out for them to face the extreme weather conditions associated with climate variations and to minimize the negative impacts of climate change (IPCC, [6]). The adaptation strategies of farmers employed to mitigate the effect of climate change include varying land size, varying the planting and harvesting dates; soil conservation techniques and mulching. Other adaptation strategies include, livestock rearing; mixed cropping, monocropping and no adaptation. Other socioeconomic adaptation measures, which help to combat the underlying causes of vulnerability and improve the adaptive capacity of farmers, must be considered. The measures taken, aimed at reducing the vulnerability of agricultural areas, depend on the resources and features of each area, as indicated so far. There will be a need to shift from certain crops to others that are more resistant and better adapted to the ‘new’ climate, or to abandon agricultural production in the more exposed areas in favor of other economic activities. In the latter case, the management of diversification off-farm needs to be managed as part of a broader economic transition. Off-farm activities involve participation in remunerative work outside the participant’s own farm and have been recognized to play an increasingly essential role in sustainable development and poverty reduction particularly in rural areas (Food and Agriculture Organization, [3]). It may serve as a safety net for the poor whereas for the rich it may be a means of accumulation. It can create opportunities to explore different adaptations strategies like change in food consumption habit, off farm activities, off farm job and temporary migration that are suitable in such changing circumstances. However, despite the growing body of knowledge attached to climate change adaptation strategies (Fasona, Tadross, Abiodun and Omojola, [4]), very few of such studies have analyzed the factors influencing adaptation in Nepalese context but there is no research on participation in off-farm activities as an adaptation strategy against climate change by farmer in the Nepal. As such, a clear understanding of the determining factors that influence farmers’ adaptation decisions is essential to the designing of appropriate policies to promote effective adaptation in the agricultural sector (Mabe, Gifty and Samuel, [9]). That might be forwarding to wider the eyes of policy maker as well as farmer who are really victimized. Several studies conducted around the globe claim that education, caste, household size, sex, age and agriculture experience of the farmers have been found determining factors related to off-farm activities (Dhungana and Khand, [1]; Dhungana, KC, Khand and Dhungana, [2]). Therefore, this study aims to identify the determinants of household adaptation strategies of climate change impact on off-farm activities in western hill of Nepal.
2. Materials and Methods

Three districts (Lamjung, Kaski and Tanahu) from hill region which are more vulnerable in terms of climatic hazards and have a wider variation in temperature and rainfall have been chosen for this study. A national adaptation program of action (NAPA) to climate change has also identified Lamjung as very high (0.787 - 1.000), and Kaski & Tanahu as moderate (0.356 - 0.600) vulnerability ranking (MoE, [10]).

For primary data collection, a multistage sampling technique was used. Initially six villages lying along a traverse across the three topographical regions in north-south direction of the western hill region were chosen from the sample districts. At the next stage, 556 households (164 out of 42048 HHs from Lamjung, and 168 out of 125459 HHs from Kaski, 224 out of 78286HHs from Tanahu districts) using proportional allocation (based on the number of households in the selected sites) were selected using systematic random sampling (CBS, 2014). The calculation of sample size was based on the assumption of about 50.0 % of people know about climate change and its impact on agriculture (CBS, 2016) with 95.0 % confidence interval and 4.5 % of margin of error which gave 475 samples. Additional 15.0 % (71 households) samples were added for addressing non-response. This gave the minimum sample of 546 for the study. For systematic random sampling, number of households for each ward was obtained from the latest population and housing census 2011. Then first household was chosen randomly starting from north-east corner of each ward. Remaining households were selected in clockwise direction at the interval of 5 households in Lamjung, 6 households in Tanahu and 4 households in Kaski districts. Finally, household head or a household member of age 45 years and above, residing in that area for the last 15 years at the time of the survey was chosen as a respondent. Only one respondent was selected from a selected household.

A rigorous literature review was done for developing research tools and was also consulted with experts for maintaining the validity. After pre-testing of 55 questionnaires; 25 in ward 33 of Pokhara metropolis of Kaski and 30 in Rupa rural municipality of Kaski districts, few modifications were made before finalizing the instrument. Information on change on food consumption habit; shifting to non-agricultural activities; shifting to non-agricultural employment; and temporary out-migration have been chosen as off-farm activities. Age, sex, marital status, educational status, caste/ethnicity, religion, household size, agriculture skill and years of experience in agricultural sector have been considered as the set of independent variables because the existing literature has also used these variables as the set of independent variables (Oppong-Kyeremen and Bannor, [11]).

For multivariate analyses, binary logistic regression models have been carried out to find the determinants of adaptation practices on off-farm activities. In binary logistic regression, the response variable contains two categories like true and false etc.

Mathematical form of binary logistic equation is given as

\[
P = \frac{1}{1 + \exp(-Z)} , \text{ where, } P = \text{estimated probability, } Z = \text{predictor}
\]

Let \( Y \) be the binary outcome variable indicating failure/success with 0/1 and \( P \) be the probability of \( Y \) to be 1, i.e. \( P = \text{prob}(Y = 1) \). Let \( X_1, \ldots, X_k \) be a set of predictor variables. Then the logistic regression of \( Y \) on \( X_1, \ldots, X_k \) estimates parameter values for \( b_0, b_1, \ldots, b_k \). For the multivariate case, \( Z \) can be expressed in terms of linear combination of other predictor variables as \( Z = b_0 + b_1X_1 + b_2X_2 + \ldots + b_nX_n \), then the logistics function reduces to the form
\[
P = \frac{1}{1 + \exp[-(b_0 + b_1 X_1 + b_2 X_2 + \ldots + b_n X_n)]}
\]

The ratio of probability of success and not success is also known as Odd ratio i.e.
\[
\text{Odds} = \frac{P}{1-P}
\]

Hence the binary logistic function reduces to the form
\[
\text{Logit}P = b_0 + b_1 X_1 + b_2 X_2 + \ldots + b_n X_n
\]

Logistic regression can also be expressed as in probability form
\[
P(x) = \frac{\exp(b_0 + b_1 X_1 + b_2 X_2 + \ldots + b_n X_n)}{1 + \exp(b_0 + b_1 X_1 + b_2 X_2 + \ldots + b_n X_n)}
\]

3. Results

Sex of the respondent, caste/ethnicity, marital status, religion, family type, family size, age, educational status, farming experience, agriculture skill and major occupation of the household are the main socio-economic and demographic variables considered for the analysis.

Nearly 2/3 of the respondents (65.1%) are male. The minimum age of the respondents is 45 years and the maximum age is 92 years with the average age as 56.55 years. More than nine-tenths (93.9%) of the respondents are married. The average family size is 5.93 which is greater than the national average (4.88). More than half (56.5%) of the respondents are Janajati followed by Brahmin (19.1%), Dalit (13.3%), Chhetri (8.3%), Muslim (1.4%), and others (1.4%). More than 2/3 (71.8%) follow the Hindu religion followed by Buddhists (22.5%), Muslim (1.4%), Christian (0.7%), and others (3.6%) respectively. Almost nine-tenths (88.3%) of the respondents do not have any agriculture skills. The minimum farming experience of the respondents is 15 years and the maximum is 70 years with a mean 31.25 years. More than 1/3 (36.0%) of the respondents have basic education followed by informal education (33.1%), illiterate (15.1%), secondary education (13.1%), and higher education (2.7%) respectively. More than nine-tenths (92.8%) of the respondents have agriculture as a major occupation followed by government services (2.3%), private sectors (1.4%), business (1.3%), and others (2.2%) respectively.

<table>
<thead>
<tr>
<th>Table 1: Background Characteristics of Respondents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Background Characteristics</td>
</tr>
<tr>
<td>Age (years)</td>
</tr>
<tr>
<td>Sex of the respondents</td>
</tr>
<tr>
<td>Male</td>
</tr>
<tr>
<td>Female</td>
</tr>
<tr>
<td>Marital status</td>
</tr>
<tr>
<td>Unmarried</td>
</tr>
<tr>
<td>Married</td>
</tr>
<tr>
<td>Divorced</td>
</tr>
<tr>
<td>Family size</td>
</tr>
<tr>
<td>Caste/ethnicity*</td>
</tr>
</tbody>
</table>
Adaptation practices toward off-farm activities

Among the various off-farm strategies adapted by the households, only four strategies: change in food consumption habits, involvement in non-agricultural activities, involvement in non-agricultural employment and migration of any household members to another place within 15 years have been considered as non-farm activities for analytical purpose. More than half of the respondents changed their food consumption habit for the adaptation of climate change. Just over one tenth of the households have shifted to non-agricultural activities and 5 % of households have shifted to non-agricultural employment. Similarly, only just over 7 % have migrated temporarily from that place for the adaptation of climate change (table 2).
Table 2: Adaptation Practices towards Off-farm Activities

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Number</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in food consumption habit</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>243</td>
<td>43.7</td>
</tr>
<tr>
<td>No</td>
<td>313</td>
<td>56.3</td>
</tr>
<tr>
<td>Involvement in non-agricultural activities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>56</td>
<td>10.1</td>
</tr>
<tr>
<td>No</td>
<td>500</td>
<td>89.9</td>
</tr>
<tr>
<td>Involvement in non-agricultural employment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>28</td>
<td>5.0</td>
</tr>
<tr>
<td>No</td>
<td>528</td>
<td>95.0</td>
</tr>
<tr>
<td>Migration of any member of HH to another place within 15 years</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>42</td>
<td>7.6</td>
</tr>
<tr>
<td>No</td>
<td>514</td>
<td>92.4</td>
</tr>
<tr>
<td>Total</td>
<td>556</td>
<td>100</td>
</tr>
</tbody>
</table>

Source: Field Survey, 2019

Determinants of households’ adaptation practices against climate change impact on off-farm activities

To find the determinants of households’ adaptation practices against climate change impact on off-farm activities, change on food consumption habit; shifting to non-agricultural activities; shift to non-agricultural employment; and temporary out-migration have been considered as dependent variables while age, sex, marital status, religion, household size, educational status, caste/ethnicity, agricultural skill and years of experience in agricultural sector are considered as independent variables.

Change of food consumption habit

Table 3 shows the adjusted odds ratio (from logistic regression analysis) with their p-values and confidence intervals (CI). From the model, analyses reveal that literate people are more likely to change the food consumption habits than illiterate people. However, with reference to upper caste, Janajati people are less likely to change of the food consumption habits. Similarly with reference to people with agriculture skill, people without agriculture skill are 2.034 times more likely to adopt such practice.

Table 3: Odds ratio from Logistic Regression Model of Change of the Food Consumption Habits (n = 556)

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Odds Ratio</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Lower</td>
</tr>
<tr>
<td>Age</td>
<td>1.004</td>
<td>0.985</td>
</tr>
<tr>
<td>Sex</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male (R)</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>1.281</td>
<td>0.891</td>
</tr>
<tr>
<td>Marital status</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Married (R)</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td>0.622</td>
<td>0.299</td>
</tr>
<tr>
<td>Religion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hindu (R)</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Non-Hindu</td>
<td>0.928</td>
<td>0.606</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Illiterate (R) 1
Literate 1.603* 1.034 2.486

**Caste/Ethnicity**
Upper Caste (R) 1
Janajati 0.469*** 0.294 0.749
Others 0.573* 0.329 0.999

**Household size**
0.991 0.929 1.058

**Agriculture skill**
Yes (R) 1
No 2.034*** 1.211 3.414

**Years of experience on agriculture**
0.986 0.967 1.005

*P <0.05, ** P <0.01 and *** P <0.001

**Note:** Final -2loglikelihood =729.281; Hosmer and Lemeshow - Chi-square value =5.655 (p = 0.686); Nagelkerke $R^2$ = 0.096; Cox-Snell $R^2$ = 0.072

**Shifting to non-agricultural activities**
Shifting to non-agricultural activities has two responses yes and no. No is taken as reference category for finding the determinants of adaptation practices. Table 4 shows the odds ratio from logistic regression analysis of shifting to non-agriculture activities with $p$-values and confidence intervals. Analyses reveal that, as household size increases, there is more likely to shift to non-agricultural activities.

**Table 4: Odds Ratio from Logistic Regression Model of Shifting to Non-agricultural Activities (n = 556)**

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Odds Ratio</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Lower</td>
</tr>
<tr>
<td>Age</td>
<td>1.008</td>
<td>0.980</td>
</tr>
<tr>
<td>Sex</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male (R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>1.247</td>
<td>0.690</td>
</tr>
<tr>
<td>Marital status</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Married(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td>0.972</td>
<td>0.274</td>
</tr>
<tr>
<td>Religion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hindu(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Non-Hindu</td>
<td>1.040</td>
<td>0.500</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Illiterate(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Literate</td>
<td>0.636</td>
<td>0.271</td>
</tr>
<tr>
<td>Caste/Ethnicity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper Caste(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Janajati</td>
<td>1.439</td>
<td>0.713</td>
</tr>
<tr>
<td>Others</td>
<td>2.527</td>
<td>0.904</td>
</tr>
<tr>
<td>Household size</td>
<td>1.143*</td>
<td>1.003</td>
</tr>
<tr>
<td>Agriculture skill</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>1.831</td>
<td>0.893</td>
</tr>
<tr>
<td>Years of experience</td>
<td>1.015</td>
<td>.983</td>
</tr>
</tbody>
</table>
on agriculture

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Odds Ratio</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.019**</td>
<td></td>
</tr>
</tbody>
</table>

Note: Final -2loglikelihood =353.552; Hosmer and Lemeshow - Chi-square value =12.717 (p = 0.122); Nagelkerke $R^2 = 0.704$; Cox-Snell $R^2 = 0.528$

*P <0.05, ** P <0.01 and *** P <0.001

Shifting to non-agricultural employment

The dependent variable shifting to non-agricultural employment has two responses i.e. yes and no assuming no as reference category. Table 5 shows the odds ratio from logistic regression analysis. As age increases, there is more likely to shift to non-agricultural employment. Female are 4.44 times more likely to shift to non-agricultural employment than male. Similarly with reference to people with agriculture skill, people without agriculture skill are 2.8 times more likely to shift into non-agricultural employment. However, with reference to married people, single people are less likely to have such adaptation practice.

**Table 5: Odds ratio from Logistic Regression Model of shifting to Non-agricultural Employment (n = 556)**

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Odds Ratio</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>1.037</td>
<td>0.998</td>
</tr>
<tr>
<td>Sex</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male (R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>4.443***</td>
<td>1.590</td>
</tr>
<tr>
<td>Marital status</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Married</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td>0.199*</td>
<td>0.056</td>
</tr>
<tr>
<td>Religion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hindu</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Non-Hindu</td>
<td>0.600</td>
<td>0.213</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Illiterate(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Literate</td>
<td>1.566</td>
<td>0.614</td>
</tr>
<tr>
<td>Caste/Ethnicity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper Caste(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Janajati</td>
<td>2.262</td>
<td>0.726</td>
</tr>
<tr>
<td>Others</td>
<td>0.538</td>
<td>0.188</td>
</tr>
<tr>
<td>Household size agriculture skill</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>2.804*</td>
<td>1.053</td>
</tr>
<tr>
<td>Years of experience on agriculture</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Note: Final -2loglikelihood =206.364; Hosmer and Lemeshow - Chi-square value =9.102 (P=0.334); Nagelkerke $R^2 = 0.850$; Cox-Snell $R^2 = 0.638$

*P <0.05, ** P <0.01 and *** P <0.001

Adapting temporary migration of any member of HH to another place within 15 years

The dependent variable; temporary migration has two responses yes and no. No has been taken as reference category for finding the determinants of adaptation practices. Table 6 shows the odds ratio of logistic regression coefficients with their p-values and 95% confidence interval. Analyses reveal that, as
household size increases, there is less likely to have adaptation practices. However, as age increases, there is more likely to have adaptation practices. Similarly, other people than Janajati are 14.78 times more likely to adapt temporary migration than upper caste. With reference to illiterate, literate people are more likely to migrate temporarily to another place.

Table 6: Odds Ratio from Logistic Regression Model of Adapting Temporary Migration of any Member to HH to another Place within 15 years (n = 556)

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Odds Ratio</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Lower</td>
</tr>
<tr>
<td>Age</td>
<td>1.013*</td>
<td>1.000</td>
</tr>
<tr>
<td>Sex</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male (R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>1.047</td>
<td>0.535</td>
</tr>
<tr>
<td>Marital status</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Married(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td>0.990</td>
<td>0.264</td>
</tr>
<tr>
<td>Religion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hindu(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Non-Hindu</td>
<td>0.943</td>
<td>0.431</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Illiterate(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Literate</td>
<td>3.838***</td>
<td>1.922</td>
</tr>
<tr>
<td>Caste/Ethnicity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper Caste(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Janajati</td>
<td>1.34</td>
<td>0.583</td>
</tr>
<tr>
<td>Others</td>
<td>14.78*</td>
<td>1.785</td>
</tr>
<tr>
<td>Household size</td>
<td>0.799***</td>
<td>0.716</td>
</tr>
<tr>
<td>Agriculture skill</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes(R)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>1.525</td>
<td>0.661</td>
</tr>
<tr>
<td>Years of experience on agriculture</td>
<td>1.013</td>
<td>0.976</td>
</tr>
</tbody>
</table>

Note: Final -2loglikelihood =275.855; Hosmer and Lemeshow - Chi-square value =10.388 (p=0.239); Nagelkerke$R^2$ = 0.786; Cox-Snell $R^2$= 0.589

*P < 0.05, ** P < 0.01 and *** P < 0.001

4. Conclusion

Most of the respondents are male of age 45-59 years, from Janajati, joint family, with basic education, Hindu, married and without agriculture skill. Religion and years of experience on agriculture does not matter for determining the households’ adaptation practices towards all selected off-farm activities. Educational status, Janajati with reference to upper caste and agriculture skill are the major determining factors for households' adaptation practices towards change in food consumption habit. Household size is only the matter for determining the households' adaptation practices for shifting to non-agricultural activities. Sex, marital status and agriculture skill matters for determining the households' adaptation practices towards shifted to non-agricultural employment. Age, Educational status, other caste than Janajati with reference to upper caste and household size are the major determining factors for households' adaptation practices towards migration to another places. The findings of this paper may be useful for preparing local adaptation plan of action in the concerned areas.
Acknowledgement

This study is fully funded by University Grants Commission (UGC) Nepal under collaborative research grants on Climate Change and Its Impact on Agriculture Sector: Evidence from western Nepal (Award No.: CRG-73/74-Ag &F-01).

References


[6] Intergovernmental Panel on Climate Change- IPCC (2013) Climate change 2013: The physical science basis; contribution of working group I to the fifth assessment report of the intergovernmental panel on climate change. Cambridge: Cambridge University Press.


On the Degree of Approximation of a Function by Nörlund Means of its Fourier Laguerre Series

Suresh Kumar Sahani\(^1\), Vishnu Narayan Mishra\(^2\) and Narayan Prasad Pahari\(^3\)

\(^1\)Department of Mathematics, MIT Campus and Rajarshi Janak Campus, T.U, Janakpur, Nepal
\(^2\)Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh, India
\(^3\)School of Mathematical Sciences, Tribhuvan University, Kathmandu, Nepal

Email: sureshkumarsahani35@gmail.com\(^1\), vishnunarayann Mishra@gmail.com\(^2\), npahari@gmail.com\(^3\)

Corresponding Author: Suresh Kumar Sahani

Abstract: In this paper, we have proved the degree of approximation of function belonging to \(L[0,\infty)\) by Nörlund Summability of Fourier-Laguerre series at the end point \(x = 0\). The purpose of this paper is to concentrate on the approximation relations of the function in \(L[0,\infty)\) by Nörlund Summability of Fourier-Laguerre series associate with the given function motivated by the works \([3],[9]\) and \([13]\).

Keywords: Nörlund methods, Fourier series, Summability.

2010 Mathematics subject Classification: 40G05, 40D05, 42A24

1. Introduction

The concepts of product summability methods are more powerful than the individual summability methods and it gives an approximation for wider class of functions than the individual methods \([6]\). A bulk number of researchers have studied the degree of approximation of a function using different summability means of its Fourier-Laguerre series \([1],[5],[7],[10],[11],[14]\) and \([15]\). This can be done at the point after replacing the continuity condition in Szegö theorem by much lighter conditions.

Let \(f(t)\) be a Lebesgue measurable function in the interval \((0,\infty)\) such that the integral

\[
\int_0^\infty e^{-x} x^\alpha f(x) L_n^{(\alpha)}(x)dx, \quad \alpha > -1
\]

exists, where \(L_n^{(\alpha)}(x)\) is the \(n\)th Laguerre polynomial of order \(\alpha > -1\) defined the generating function

\[
\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \omega^n = (1 - \omega)^{-\alpha - 1} e^{\frac{-x\omega}{1 - \omega}}
\]

The Fourier series of the function \(f(x)\) is given by

\[
f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \quad \alpha > -1,
\]

where the coefficients \(a_n\) are the defined by the following formulae

\[
\Gamma(\alpha + 1) A_n^\alpha = \int_0^\infty e^{-x} x^\alpha f(x) L_n^{(\alpha)}(x)dx
\]
where \( A_n^\alpha = \left( \frac{n + a}{n} \right) \sim n^\alpha \).

Also, we write \( \phi(x) = \frac{1}{\Gamma(a+1)} e^{-x} x^\alpha \)  \hspace{1cm} (1.5)

Let \( \Sigma u_n \) be a given infinite series with the sequence \( \{S_n\} \) of its partial sums. Let \( \{p_n\} \) be a sequence of real or complex constants with \( p_n \) as its non-vanishing \( n \)th partial sum. The sequence to sequence transformation is given by

\[ \tau_n = \frac{1}{p_n} \sum_{m=0}^{n} p_{n-m} S_m. \]  \hspace{1cm} (1.6)

which defines the sequence of Nörlund means of the sequence \( \{S_n\} \), generated by the sequence of coefficients \( \{p_n\} \). If \( \tau_n \to s \) then \( \Sigma u_n \) or the sequence \( \{S_n\} \) to the sum \( S \). If the sequence of Nörlund mean \( \{\tau_n\} \) is of bounded variation i.e. if

\[ |\tau_n - \tau_{n-1}| < \infty \]  \hspace{1cm} (1.7)

then the series \( \Sigma u_n \) is said to be absolute Nörlund summable \([5, 6]\) and \([12]\). The absolute Nörlund summability of a Fourier series has been studied by several researchers like Bor \([5, 7]\), Mazhar \([10]\), Fadden \([11]\), Padhy, Tripathi and Mishra \([14]\) and Siddiqui, Gi, Bohar and Brono \([15]\).

In 1976, Yadav \([19]\) was the first to establish a result on absolute Nörlund summability of Laguerre series at origin and in 1979, Beohar and Jadiya \([2]\) established a result on the absolute Nörlund summability of Laguerre series at the point \( x = 0 \). Later on, several researchers like Alghamdi & Mursaleen \([1]\), Beohar and Jadiya \([2, 3]\), Khatri and Mishra \([8]\), Karasniqi \([9]\), Nigam and Sharma \([13]\), Shanker \([16]\), Tiwari and Kachhara \([18]\) obtained the degree of approximations of \( L[0, \infty) \) of the Fourier-Laguerre series by Cesaro mean Nörlund, Euler, \((C,1)(E,q),(C,2)(E,q)\) harmonic-Euler mean et al. Concerning the absolute Nörlund summability of Laguerre series Yadav \([19]\) has proved the following theorem:

**Theorem 1.1:** If \( \{p_n\} \) be a non-negative and non-increasing sequence of constant such that

\[ p_n = \sum_{r=0}^{n} p_r \to \infty \]  \hspace{1cm} and \[ \sum_{n} \frac{(-1+2\alpha)}{4 p_n} \] is convergent. i.e. \[ \sum_{n} \frac{(-1+2\alpha)}{4 p_n} < \infty \], then the Fourier series (3) is \( |N, p_n| \) summable at the end point \( x = 0 \), provided that for some small positive \( \epsilon \),

\[ \phi(x) \in BV[0, \infty) \]  \hspace{1cm} (1.8)

where

\[ \int_{0}^{\infty} e^{-\alpha} x^a \frac{7}{12} \ |f(x)| dx = 0 \left( n^{-1} \right) \] and \( F(x) \) is bounded in \([0, \epsilon]\).  \hspace{1cm} (1.9)

Where,

\[ \phi(x) = [f(x) - f(0)] e^{-x} x^\alpha \]  \hspace{1cm} (1.10)

and,

\[ F(x) = [f(x) - f(0)] e^{-x} x^\frac{(-1+2\alpha)}{4} \]  \hspace{1cm} (1.11)
2. Main Results

Before proceeding with the main result, we begin with recalling some Lemmas that are required to prove main theorem in this paper.

**Lemma 2.1:** (Szegö [17], p.177) Let \( \alpha \) be arbitrary and real, \( c \) and \( w \) be fixed positive constants, and let \( n \to \infty \). Then

\[
L_n^{(\alpha)}(x) = \begin{cases} 
\frac{x^{\frac{\alpha+1}{4}}}{n^{\frac{\alpha-1}{4}}}, & \text{if } \frac{c}{n} \leq x \leq w \\
o(n^\alpha), & \text{if } 0 \leq x \leq \frac{c}{n} 
\end{cases} \quad (2.1)
\]

**Lemma 2.2:** (Szegö [17], p. 240) Let \( \alpha \) be arbitrary and real, \( w > 0 \), \( 0 < \eta < 4 \). Then we have for \( n \to \infty \),

\[
\max e^{\frac{-x}{2}}.x^{\frac{\alpha+1}{4}}, |L_n^{(\alpha)}(x)| \sim \begin{cases} 
\frac{x^{\frac{\alpha+1}{4}}}{n^{\frac{\alpha-1}{4}}}, & \text{if } w \leq x \leq (4 - \eta).n \\
\frac{x^{\frac{\alpha-1}{12}}}{n^{\frac{\alpha-1}{2}}}, & \text{if } x \geq w 
\end{cases} \quad (2.2)
\]

**Lemma 2.3:** (Bhatta [4]) Let \( \sum a_n \) be an infinite series with \( S_n \) as its \( n^{th} \) partial sum and \( \{p_n\} \) be a non-increasing sequence such that \( p_n \to \infty \). If \( \sum_n \frac{|a_n|}{p_n} < \infty \) i.e. convergent, then the series \( \sum_n a_n \) is \([N,P_n]\) summable.

Considering the above facts, we prove the following theorem:

**Theorem 2.4:** Let \( \chi(x) \) be a non-negative, and non-increasing function of \( x \) such that \( x^\alpha \cdot \chi \left( \frac{1}{x} \right) \to 0 \) as \( n \to 0 \). Let \( \{p_n\} \) be a non-negative and monotonic non-increasing sequence of constants with \( P_n \) as its non-vanishing \( n^{th} \) partial sum such that \( \sum_n \frac{|\chi(n)|}{p_n} < \infty \) i.e. convergent. Let \( -\frac{1}{2} > \alpha > -1 \) and \( w \) is a fixed positive constant. If

\[
\int_x^w \frac{\phi(s)}{S^{\alpha+1}} ds = o\left[ \chi \left( \frac{1}{x} \right) \right], \text{ as } x \to 0, \quad (2.4)
\]

\[
\int_x^w e^{\frac{-x}{2}}.x^{\frac{-\alpha}{4}} \cdot \frac{3}{4} |\phi(x)| dx = o\left[ n^{\frac{-\alpha-1}{4}} \cdot \chi(n) \right] \quad (2.5)
\]

and

\[
\int_x^\infty e^{\frac{-x}{2}}.x^{\frac{-(6\alpha+7)}{12}} |\phi(x)| dx = o\left[ n^{\frac{-(2\alpha+1)}{4}} \chi(n) \right] \text{ as } n \to \infty, \quad (2.6)
\]

then the Fourier-Laguerre series (1.3) is \([N,P_n]\) summable at the point \( x = 0 \).

**Proof:**

First of all, under the hypothesis (2.4), we prove

\[
\int_0^x |\phi(s)| ds = o\left[ x^{\alpha+1} \cdot \chi \left( \frac{1}{x} \right) \right] \quad (2.7)
\]

For this, we have

\[
I(s) = \int_x^w \phi(s) ds = o\left[ \chi \left( \frac{1}{x} \right) \right]
\]

Then

\[
|\phi(s)| = -I'(s) \cdot S^{\alpha+1}
\]

Therefore,

\[
\int_0^x |\phi(s)| ds = -\int_0^x S^{\alpha+1} \cdot I'(s) ds
\]

\[
= -[S^{\alpha+1} \cdot I(S)]^x_0 + (\alpha + 1) \int_0^x S^\alpha \cdot I(s) ds
\]

67
Suresh K. Sahani, Vishnu N. Mishra, Narayan P. Pahari/On The Degree of Approximation of by…

\[ = o \left[ S^{n+1} \cdot \chi \left( \frac{1}{3} \right) \right] + o \left[ S^{n+1} \cdot \chi \left( \frac{1}{3} \right) \right] \]

Then the proof of the theorem is as follows:

Since \( l_n^{(\alpha)} (0) = \binom{n+\alpha}{\alpha} \) (2.8)

therefore \( S_n(0) = \sum_{k=0}^{n} a_k \cdot L_k^{(\alpha)} (0) \)

\[ = \sum_{k=0}^{n} \frac{1}{\Gamma(\alpha+1)} \int_{0}^{\infty} e^{-x} \cdot x^\alpha f(x) \sum_{k=0}^{n} L_k^{(\alpha)} (x) dx \]

\[ = \frac{1}{\Gamma(\alpha+1)} \int_{0}^{\infty} e^{-x} \cdot x^\alpha f(x) L_n^{(\alpha+1)} (x) dx \]

Again, due to the orthogonality of Laguerre polynomials, we have

\[ S_n (0) - f (0) = \int_{0}^{\infty} \phi(x) L_n^{(\alpha+1)} (x) dx \]

\[ = \left( \int_{0}^{c} + \int_{c}^{w} + \int_{w}^{n} + \int_{n}^{\delta} \right) \phi(x) L_n^{(\alpha+1)} (x) dx \]

\[ = I_1 + I_2 + I_3 + I_4 \text{ (say)} \] (2.9)

First we consider \( I_1 \),

\[ |I_1| = \int_{0}^{c} |\phi(x)| \cdot L_n^{\alpha+1} (x) dx \]

\[ = o(n^{\alpha+1}) \int_{0}^{c} |\phi(x)| dx \]

\[ = o[\chi(n)], \text{ as } n \to \infty. \] (2.10)

Next we consider \( I_2 \), using (2.1) and (2.4), we have

\[ |I_2| = o \left( \frac{n^{\alpha+1}}{2^{\alpha+1}} \right) \int_{c}^{w} \frac{\phi(x)}{x^{\alpha+1}} |x^{\alpha+1} dx \]

\[ = o \left( \frac{n^{\alpha+1}}{2^{\alpha+1}} \right) \int_{c}^{w} \frac{\phi(x)}{x^{\alpha+1}} \left( x^{\alpha+1} \right) dx \]

\[ = \left( \frac{n^{\alpha+1}}{2^{\alpha+1}} \right) o \left( \frac{n^{\alpha+1}}{2^{\alpha+1}} \right) \int_{c}^{w} \frac{\phi(x)}{x^{\alpha+1}} dx \]

\[ = o(1) o[\chi(n)] \]

\[ = o[\chi(n)], \text{ as } n \to \infty. \] (2.11)

Considering \( I_3 \) and using (2.2), (2.3) and (2.5), we have

\[ |I_3| = \int_{w}^{n} |\phi(x)| \left| L_n^{(\alpha+1)} (x) \right| dx \]

\[ = \int_{w}^{n} |\phi(x)| \left| e^{-x} x^{-\frac{(2\alpha+3)}{4}} \phi \left( n^{\frac{2\alpha+1}{4}} \right) \right| dx \]

\[ = o \left( n^{\frac{2\alpha+1}{4}} \right) \int_{w}^{n} e^{-x} x^{-\frac{(2\alpha+3)}{4}} |\phi(x)| dx \]
\[= o \left( n^{-\frac{2n+1}{4}} \right) o \left( n^{-\frac{(2n+1)}{4}} \chi(n) \right)\]
\[= o[\chi(n)], \text{ as } n \to \infty \quad (2.12)\]

Again, considering \( L_4 \), using (2.2), (2.6) and (2.7), we have
\[|L_4| = \int_{n}^{\infty} |\phi(x)| \frac{x}{n} e^{\frac{x}{n}} \left( n^{-\frac{(2n+1)}{4}} \chi(n) \right) \text{ dx}\]
\[= o \left( n^{-\frac{6n+5}{12}} \right) \int_{n}^{\infty} e^{\frac{x}{n}} \frac{x}{n} \left( n^{-\frac{(2n+1)}{4}} \chi(n) \right) \text{ dx}\]
\[= o \left( n^{-\frac{6n+5}{12}} \right) \int_{n}^{\infty} e^{\frac{x}{n}} \frac{x}{n} \left( n^{-\frac{(2n+1)}{4}} \chi(n) \right) \text{ dx}\]
\[= o \left( n^{-\frac{(2n+1)}{4}} \chi(n) \right), \text{ as } n \to \infty \quad (2.13)\]

Now combining (2.10), (2.11), (2.12) and (2.13), and using \( f(0) = 0 \), we get
\[|S_n(0) - 0| = o[\chi(n)], \text{ as } n \to \infty \text{ and therefore } |S_n| = o[\chi(n)], \text{ as } n \to \infty.\]

Therefore applying (2.7), we have
\[\sum_{n} \frac{|S_n|}{p_n} = \sum_{n} \frac{\chi(n)}{p_n} < \infty \quad \text{i.e. convergent}.\]

This completes the proof of the theorem.

**Conclusion**

In this paper, we have proved a theorem related to the degree of approximation of function belonging to \( L[0, \infty) \) by Nörlund Summability of Fourier-Laguerre series at the end point \( x = 0 \). This work establishes some of the results that characterize the approximation relations of the function in \( L[0, \infty) \) by Nörlund Summability of Fourier-Laguerre series. In fact, these results can be used for further study in many practical problems in science and engineering.

**References**


Basic Operations on Vedic Mathematics: A Study on Special Parts

Krishna Kanta Parajuli
Department of Mathematics, Valmeeki Campus, Nepal Sanskrit University
Email: kknmparajuli@gmail.com

Abstract: Vedic Mathematics was rediscovered and reconstructed by Sri Bharati Krishna Tirthaji from ancient Sanskrit texts Veda early last century between 1911 – 1918 is popularly known today is Vedic Mathematics. It is an extremely refined, independent and efficient mathematical system based on his 16 formulae and some sub-formulae with simple rules and principles. The main purpose of this paper is to communicate a new approach to Mathematics, offering simple, direct, one-line, mental solutions to mathematical problems. In the way of basic mathematical operations like addition, subtraction, multiplication and division can be done in simple ways, and results are obtained quickly and can be checked in a minute by using the Vedic techniques. In this system, for any problem, there is always one general technique and also some special pattern problems. This paper especially concentrates only on the specific pattern of elementary operation of Vedic Mathematics.

Keywords: Vedic Mathematics, Ekadhikena Purvena, Nikhilam Navataschara Dasatah, Urdhvatiryagbhya, Antyayordashkepi.

1. Introduction

The most common meaning of Veda is knowledge [3]. It is considered as the oldest layer of Sanskrit literature and the oldest scriptures of Hinduism. The Vedas are considered divine in origin and are assumed to be direct revelations from God [7]. There are four Vedas: Rigveda, Yajurveda, Samaveda and Atharvaveda [2]. The Vedas are ancient writings whose date is disputed but which date from at least several centuries of B.C. The content of the Vedas was known long before writing was invented and was freely available to everyone. It was passed on by word of mouth. The writings called the Vedas to consist of a huge number of documents (there are said to be millions of such documents, many of which have not yet been translated) and these have recently been shown to be highly structured, both within themselves and concerning each other [2].

Bharati Krishna Tirthaji spent eight years between 1911 – 1918 at the Shringari forest near Shringari Moth for the practice of Brahma-Sadhana to study the advanced Vedanta Philosophy [10]. According to him, the rediscovery and reconstruction of Vedic Mathematics were one of the outputs of his devotion from stray references within the appendix portions of the Atharvaveda. Vedic Mathematics is a system of reasoning and mathematical working based on 16 formulas and 13 sub-formulas with simple rules and principles. Vedic mathematical techniques are also based on the ancient mathematical system as well as the modern system. Each formula provides a principle of mental working applicable to many diverse areas of Mathematics [10].

The most significant quality of Vedic Mathematics is its consistency. It sharpens the mind, improve memory power with concentration, speed up mathematical calculations, minimize careless mistakes and encourage innovations [12]. The beautiful coherence between arithmetic and algebra is visible in the Vedic system. The real beauty and effectiveness of Vedic Mathematics cannot be fully appreciated without actually practicing the system [14]. It will be benefitted from these wonderful, logical, and systematic and faster methods of
solving the most complex sums. It helps the students to remember that many big digit calculations can be done much faster by Vedic methods than the calculator [5].

After going through the content presented in this paper, serious mathematical issues, higher-level mathematical problems are not taken up in this paper, even though many aspects like four fundamental operations: addition, subtraction, multiplication and division are operated with. In Vedic Mathematics, there are two types of techniques: Specific and general. Those techniques which are fast and effective but can be applied only to a particular combination of numbers are called specific and those which have a much wider scope of application than specific as they deal with a wider range of numbers are general. Mathematical calculations can be done much faster by Vedic methods than by calculators when we use the specific methods. So, this article is concentrated only on the tiny glimpse of elementary operations on mathematics by specific techniques of Vedic Mathematics.

2. Basic Operations on Vedic Mathematics

In Sanskrit, the terms sutra means ‘Thread of Knowledge’. Vedic Mathematics consists of 16 sutras and a similar number of sub-sutras. Only some of these formulae are used in the basic operation of arithmetic. The meaning of these used sutras in this article is tabulated below [6], [9], [10].

<table>
<thead>
<tr>
<th>Vedic Sutras (Formulae)</th>
<th>Meanings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ekadhikeana Purvena</td>
<td>By one more than the previous one</td>
</tr>
<tr>
<td>Nikhilam Navatascaramam dasatah</td>
<td>All from nine and last from ten</td>
</tr>
<tr>
<td>Urdhvatiryagbhhyam</td>
<td>Vertically and crosswise</td>
</tr>
<tr>
<td>Sankalana-Vyavakalanabhhyam</td>
<td>By addition and subtraction</td>
</tr>
<tr>
<td>Puranapuranabhyam</td>
<td>By completion and non-completion</td>
</tr>
<tr>
<td>Paravartya Yajaayet</td>
<td>Transpose and apply</td>
</tr>
<tr>
<td>Ekanyunena Purvena</td>
<td>By one less than the previous one</td>
</tr>
<tr>
<td>Antyayordashkepi</td>
<td>The total of the last digit is ten and the previous part is the same</td>
</tr>
<tr>
<td>Antyayoshatakepi</td>
<td>The total of the last digit is a hundred and the previous part is the same</td>
</tr>
<tr>
<td>Vamanlyayoh Dashakepi</td>
<td>The total of the last digit is ten and the previous part is the same</td>
</tr>
<tr>
<td>Vamanlyayoh Dashake Gunijah Api</td>
<td>The total of left two digits is a multiple of ten and the unit digit is the same</td>
</tr>
</tbody>
</table>

2.1 Addition

The formula used for addition in Vedic Mathematics is Puranapuranabhyam, Sankalana-Vyavakalanabhhyam and Ekadhikeana Purvena. The whole procedure of adding can be summarized in the following steps [8], [9]:

Add the digits column-wise and when the running total becomes greater than 10, put a dot or tick on that number. Move ahead with the excess of 10 and add it to the next digit of the column. Lastly, count the number of dots or ticks and add it to the next column.

Examples: 

| 9’8’7 | 4 2 0 0 |
| 4 6’6’ | 0 0 9’8’ |
| 8’4’7’ | 8’ 7’6’5’ |
| + 2 4 8 | +5 7 8 9 |
| 2 5 4 8 | 1 8 8 5 2 |
2.2. Subtraction

For the subtraction process, we use the Vedic formula Nikhilam Navatascaramam dasatah. The meaning of this formula is 'All from nine and the last from ten'. This method works faster when subtraction is done from a multiple of 10 i.e. 10, 100, 1000, 10000, … . While calculating is adopted by the conventional method, several carry-overs are needed, which wastes time and confusion about accuracy remains, the Vedic method helps us in this regard and saves our precious time [4], [13]. The concepts can be illustrated by taking examples as following:

- Start moving from right to left. Replace every zero from the left with a 9 and the last zero with a 10. The extreme left digit before zero will get reduced by 1 [8], [9].

  For example, for subtracting 5472 from 400000
  
  \[
  \begin{array}{c}
  400000 \\
  -5472 \\
  \hline
  \end{array}
  \]
  
  will become

  \[
  \begin{array}{c}
  399910 \\
  -5472 \\
  \hline
  394528
  \end{array}
  \]

- When the digit at minuend (upper digit) > Subtrahend digit (lower digit), normal subtraction is done.
- In case the upper digit < lower digit, we take the complement of the difference (i.e. complement of 0 is 10. Complement of 1 is 9, the complement of 2 is 8 and so on). The complement of the last digit is taken from 10 and the complements of the rest of the digits are taken from 9.
- When we arrive at a stage where there is no need to take the complement, subtract 1 extra from that column.

  For example, to subtract, 89543 – 40597, we write

  \[
  \begin{array}{c}
  8 9 5 4 3 \\
  -4 0 5 9 7 \\
  \hline
  4 8 9 4 6
  \end{array}
  \]

  from 10

2.3 Multiplication

For multiplication, we can use eight Vedic formulae Antyayordashakepi, Nikhilam Navatascharam Dasatah, Anurupena, Ekanyunena Purvena, Antyayoshatakepi, Vamanlayoh Dashakepi, Vamanlayoh Dashake Gunijah Api, Urddhvatiryagbhyam [9] [10].

Except for the last formula Urddhvatiryagbhyam, all are the specific formula of multiplication, which has limited application. Many special formulae help us to find the answer to a special type of multiplication even in seconds and the Urddhvatiryagbhyam method helps us to encounter all types of multiplication. The special types of formulae are:

2.3.1 Antyayotdashakepi

This formula has limited application and is valid as long as the sum of the unit digit at multiplicand and multiplier is 10 and the remaining digits are the same. The final product will consist of two parts LHS and RHS [10], [12].

Multiply the unit digits and write it in the RHS part. In the LHS part, write the product of (Remaining digit at Ten's/hundred place) \( \times \) (Remaining digit at Ten's/hundred place +1).

For example, to multiply 75 by 75, RHS = \( 5 \times 5 = 25 \) and LHS = \( 7 \times (7 + 1) = 72 \) (by using the formula Ekadhiakena Purvena). So, result = 7225.

2.3.2 Vamanlayoh Dashakepi

This formula is applicable when the sum of digits placed at the ten's place in the multiplicand and multiplier is 10 and the unit digit of both multiplicand and multiplier is the same. For method, see, [8], [1].

The answer to such questions consists of two parts.
LHS = Product of two left digits whose sum is 10 + unit digit; and RHS = Square of the unit digit.
For example, to multiply 98 by 18, LHS = 9 \times 1 + 8 = 17 and RHS = 8 \times 8 = 64. The result = 1764.

2.3.3. Nikhilam Navatascharam Dasatah

This formula works better when both the multiplicand and multiplier are very close to the base. The base should be in the form of 10^n, where n is a natural number. The ideas can be illustrated in stepwise as below [10].
- Write the two numbers one below the other and write the deviations of the two numbers from the base.
- There are two parts
  (a) the left-hand part will be obtained by cross operation of two numbers written diagonally.
  (b) the right side of the answer will be obtained by multiplying the deviations.
- The number of digits in the right-hand part will be in accordance with the number of zeros in the base number.

Under this formula, there are many cases

(i) When both the numbers are above the base:

For examples, the multiplication of 104 by 103 and 16 by 12 are expressed as:

\[
\begin{align*}
104 + 4 & \quad (\text{where the base is } 100) \\
103 + 3 & \\
\hline
107 / 12 & \\
\text{So, result} = 10712
\end{align*}
\]

\[
\begin{align*}
16 + 6 & \quad (\text{where the base is } 10) \\
12 + 2 & \\
\hline
18 / 12 & \\
\text{So, result} = 192
\end{align*}
\]

(ii) When both the numbers are below the base

For example, to multiply 97 by 98, it is expressed as

\[
\begin{align*}
97 & \quad 3 \\
98 & \quad 2 \\
\hline
95 / 06 & \\
\text{(where the base is } 100) \\
\text{So, result} = 9506
\end{align*}
\]

(iii) When one number is above the base and another is less than the base.

To multiply 15 by 9, it can be expressed as:

\[
\begin{align*}
15 + 5 & \\
9 & \quad 1 \\
\hline
14 / -5 & \\
14/10 - 5 \rightarrow 13/5 & \text{(where the base is } 10) \\
\text{So, result} = 135. \text{ It is noted that when there is a (–) sign at the right-hand side, we use the Nikhilam formula i.e. subtracting the right-hand digit (–5) from 10 and the left-hand part will get diminished by 1.}
\end{align*}
\]

2.3.4. Ekanyunena Purvena [9], [10]

The meaning of the Formula is "By one less than the previous one". This formula is used when the multiplier is 9 or 99 or 999 or 9999 etc. The method is divided into two cases

(i) When the number of digits in the multiplicand is equal or less to the number of nines:

The method is as: Subtract 1 from the multiplicand and write the result in LHS and subtract the multiplicand by applying the Nikhilam formula and write the result in RHS.

For example, to multiply 3785 by 999999, we have

\[
\begin{align*}
\text{LHS} = 3785 - 1 = 3784, \quad \text{RHS} = 999999 - 3784 = 996215, \quad \text{Result} = 3784996215.
\end{align*}
\]
(ii) When the number of digits in the multiplicand is higher than the number of nines:

It is a little different from (i). To get the result, we have to

- Add as many zero as the numbers of nines to the multiplicand.
- Subtract the original multiplicand from the figure obtained in the 1\textsuperscript{st} step.

For example, to multiply 23758 by 999, there are 3 nines, so for this, subtracting original multiplicand:

\[23758000 - 23758 = 23734242.\]

In the above discussion of case (ii), the formula Ekanyunena Purvena is not seen in use, but can also be used.

2.4. Division

In division operation, we shall deal with three Vedic formulae: \textit{Nikhilam}, \textit{Paravartya Yojayet} and \textit{Urdhvatiryagbhyam} [10]. Here, \textit{Nikhilam} and \textit{Paravartya Yojayet} are specific rules whereas \textit{Urdhvatiryagbhyam} is a general rule which is also known as the Dhawajanka method of division and is based on the long-established Vedic process of mathematical calculations [10], [12].

In the \textit{Nikhilam} formula, it has limited application and is useful when every digit of the divisor is greater than 5. The best part of this formula is that there is no subtraction process to be carried out at all [1], [9].

\textit{Paravartya Yojayet} formula is Transpose and Apply, which is slightly different from the \textit{Nikhilam} formula. \textit{Paravartya Yojayet} formula works effectively when the first digit of the divisor is 1. In the \textit{Paravartya} formula, the complement obtained from the \textit{Nikhilam} formula will be revised by changing the sign separately. For example, if the divisor is 86, the nearest base = 100 and complement = 100 – 86 = 14 and revised complement = –1 – 4.

For the \textit{Nikhilam} method, the working rules are [8], [9], [10].

- Take a base (in the power of 10) nearest to the divisor and write its complement below the original divisor.
- Separate the extreme right digit of the divided by drawing a slash equal to the number of digits in the divisor. This block is known as the remainder block and the left block is known as the quotient block.
- The number of digits to be placed in the remainder column should be equal to the number of zeros in the base.
- Carry down the first digit of the divisor, which will be the first digit of the quotient, multiply this quotient by the complement and place it in the dividend column; next to the first digit of the dividend.
- Write mechanically the sum of the digits of the 2\textsuperscript{nd} column to get the 2\textsuperscript{nd} digit of the quotient.
- Repeat the process until we get a number in the remainder column. If the remainder is greater than the divisor, continue the same process in the remainder block until the digit in the remainder column is less than that of the original divisor.

For example, to divide 1221340 by 8987,

<table>
<thead>
<tr>
<th>Divisor Column</th>
<th>Quotient Column</th>
<th>Remainder Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>8987</td>
<td>1 2 2</td>
<td>1 3 4 0</td>
</tr>
<tr>
<td>Complement = 1013</td>
<td>1 0</td>
<td>1 3</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0 3 9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5 0 6 5</td>
</tr>
<tr>
<td></td>
<td>1 3 5</td>
<td>8 0 9 5</td>
</tr>
</tbody>
</table>

Here, quotient = 135 and remainder = 8095.

3. Findings

The documents for the special parts of basic operation on Vedic Mathematics may not be sufficient for the reader. But the presented documents and its representative examples are clear enough to obtain some
findings with its techniques. There is a contrast between Vedic and Conventional Methods for calculating basic operations. Vedic Mathematics itself is the easiest, enjoyable and one-line mental form of mathematics and some of the calculations are faster than the calculator. Modern methods have just one way of doing, say, division and this is so cumbersome and tedious that the students are now encouraged to use a calculating device whereas the Vedic method can be done without devices. While calculating is adopted by the conventional method, several carry-overs in subtractions, large multiplication tables in multiplications and many hits and trial methods are needed, which wastes time and confusion about accuracy remains, the Vedic method helps us in this regard and saves our precious time.

4. Conclusion
Vedic Mathematics is considered as mental Mathematics. It develops accuracy, exactness and precision. The real beauty and effectiveness of Vedic Mathematics cannot be fully appreciated without actually practicing the system. Hence the methods discussed, and organization of the content of the paper shows the basic operation of Vedic Mathematics is an extremely refined and efficient mathematical system. The technique involved in basic operations of Vedic Mathematics is highly efficient, and certainly needs an explanatory approach for further development. This article hence is an explanatory approach for the basic mathematical operations using Vedic Mathematics; its special parts. So, a stepwise, procedural and algorithmic framework of Vedic Mathematics in basic operations can be drawn from this article.

5. Suggestions and Implications
With the knowledge of Vedic Mathematics at primary level classes, mathematics would become a favorite subject of all, as they would be able to perform calculations accurately with speedily. To realize this objective, the assimilation of Vedic Mathematics should be given prime importance. The methods and principles can be integrated into an existing school curriculum. The incorporation of Vedic Mathematics into the present issue-based approach makes the system both conceptual and calculation based. Both Vedic Mathematics and Conventional Mathematics give the same result on calculation. Students should be trained on both methods and they should be given to choosing between the methods which they find convenient.

References

A Note on Natural Transformation of Exton’s Triple Series Hypergeometric Function

Harsh vardhan Harsh¹, Puneet Krishna Sharma² and Shabana Khan³

¹Faculty of Sci. & Tech., ICFAI Tech. School, ICFAI University Jaipur, Agra Road, Jamdoli, Jaipur, Rajasthan, India
²Department of Applied Mathematics, Amity School of Applied Sciences, Amity University Jaipur, Rajasthan, India
³Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia University Okhla, New Delhi, India

E-mail: ¹harshvardhanharsh@gmail.com, ²puneetkrishnadps@gmail.com, ³areenamalik30@gmail.com

Corresponding Author: Puneet Krishna Sharma

Abstract: Triple series hypergeometric functions are very important from the applications point of view. Exton had defined 20 triple hypergeometric functions namely \( X_1, X_2, \ldots, X_{20} \). Integral transform technique is widely using for research purpose. Natural transformation is a new kind of integral transform and generalization of Laplace transform. In the present research note, we give the Natural integrals of the triple series hypergeometric function due to Exton.

Keywords: Hypergeometric functions, Exton’s triple series, N-transform.

1. Introduction

Triple series hypergeometric functions are well known in literature and large number of applications are available. Srivastava and Kashyap [11] present several interesting applications of hypergeometric series in one and more variable in queuing theory and stochastic process. Lauricella [8] introduced 14 complete set of hypergeometric series in three variables. Saran [9] initiated systematic study of Lauricella set. Srivastava and Karlsson [10] give detailed account on triple series in his famous monograph. In 1982 Exton [2] had defined 20 triple functions namely \( X_1, X_2, \ldots, X_{20} \) in his famous note. In the same note, Laplace integrals of Exton’s series [7] and wide applications are also given which shows importance and applicability of the functions. Integral Transform technique is an important and frequent using tool in mathematical and computational research. Recently developed Natural transform is widely using for application purpose. The natural transformation [3], initially defined by khan and khan as N-transform, who studied properties and applications. Belgacem et. al. [6],[1] defined its inverse and studied some additional fundamental properties of this transform and named it the Natural transform.

2. Results Required

The Natural transform [5],[6] \( R(s,u) \) of the function for all \( t \geq 0 \), is given by

\[
N^+ \left[ f(t) \right] = R(s,u) = \frac{1}{u} \int_0^\infty e^{-st} f(ut) dt \quad s > 0, u > 0
\]

\[
R(s,u) = \frac{1}{u} \int_0^\infty e^{-ut} f(t) dt,
\] (2.1)
where \( t, u \) are time variables and \( s \) is the frequency variable, provided the function \( f(t) \) is defined in a set
\[
A = \left\{ f(t) : \exists \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{\tau_1}; t \in (-1)^j \times [0, \infty) \right\}, \tag{2.2}
\]

where \( M \) is finite constant, \( \tau_1, \tau_2 \) may be finite or infinite. The discrete form of natural transform \([4]\) is given by
\[
N^+ \left[ f(t) \right] = R(s,u) = \sum_{n=0}^{\infty} \frac{n!a_nu^n}{s^{n+1}} \tag{2.3}
\]
The inverse Natural transformation is defined by
\[
N^{-1} [R(s,u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} R(s,u) ds \tag{2.4}
\]

By using these definitions, shifted factorial is defined as follows:
\[
(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = s^u \int_0^{\infty} e^{st} \left( \frac{st}{u} \right)^{a+m-1} dt
\]

Exton has defined following 20 triple series hypergeometric functions:
\[
X_1(a,b;c,d;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p} (b)_p x^m y^n z^p}{(c)_m (d)_n p! m! n! p!} \tag{2.5}
\]
\[
X_2(a,b;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p} (b)_p x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!} \tag{2.6}
\]
\[
X_3(a,b;c,d;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b)_p x^m y^n z^p}{(c)_m (d)_n p! m! n! p!} \tag{2.7}
\]
\[
X_4(a,b;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b)_n p! x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!} \tag{2.8}
\]
\[
X_5(a,b_1,b_2;c;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b_1)_n (b_2)_p x^m y^n z^p}{(c)_m (d)_n p! m! n! p!} \tag{2.9}
\]
\[
X_6(a,b_1,b_2;c,d;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b_1)_n (b_2)_p x^m y^n z^p}{(c)_m (d)_n p! m! n! p!} \tag{2.10}
\]
\[
X_7(a,b_1,b_2;c,d;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b_1)_n (b_2)_p x^m y^n z^p}{(c)_m (d)_n p! m! n! p!} \tag{2.11}
\]
\[ X_n(x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b)_{n+p} (c)_{m+n+p} x^m y^n z^p}{m! n! p!} \]
3. Main Results

The following twenty interesting integrals involving generalized hypergeometric function will be evaluated in this paper:

1. \( X_1(a,b;c,d;x,y,z) = \)

\[
\frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(u+v)} \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \phi_1 \left( -c; x; \frac{xs^2}{u^2} \right) \phi_1 \left( -d; y; \frac{ys^2}{u^2} + \frac{z}{uv} \right) dsdt \]  

(3.1)

2. \( X_2(a,b;c_1,c_2,c_3;x,y,z) = \)

\[
\frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(u+v)} \left( \frac{s}{u} \right)^{a-1} \phi_1 \left( -c_1; x; \frac{xs^2}{u^2} \right) \phi_1 \left( -c_2; \frac{ys^2}{u^2}; \frac{zs}{u} \right) dsdt \]  

(3.2)

3. \( X_3(a,b;c,d;x,y,z) = \)

\[
\frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(u+v)} \left( \frac{s}{u} \right)^{a-1} \phi_1 \left( -c; x; \frac{xs^2}{u^2} + \frac{ys^2}{u^2} \right) \phi_1 \left( -d; \frac{zs}{u}; \frac{zs}{uv} \right) dsdt \]  

(3.3)

4. \( X_4(a,b;c_1,c_2,c_3;x,y,z) = \)

\[
\frac{1}{\Gamma(a)} \int_0^\infty e^{-(u)} \left( \frac{s}{u} \right)^{a-1} \phi_1 \left( -c_1; x; \frac{xs^2}{u^2} \right) y^2 \phi_2 \left( b;c_2,c_3; \frac{ys}{u}, \frac{zs}{u} \right) ds \]  

(3.4)

5. \( X_5(a,b_1,b_2;c;x,y,z) = \)

\[
\frac{1}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)} \int_0^\infty \int_0^\infty e^{-(u+v)} \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b_1-1} \phi_1 \left( -c; x; \frac{xs^2}{u^2} + \frac{ys}{u^2} + \frac{z}{uv} \right) dsdt \]  

(3.5)

6. \( X_6(a,b_1,b_2;c,d;x,y,z) = \)

\[
\frac{1}{\Gamma(a)\Gamma(b_2)} \int_0^\infty \int_0^\infty e^{-(u+v)} \left( \frac{s}{u} \right)^{a-1} \phi_1 \left( -c;b_2;d; \frac{zs}{u} \right) dsdt \]  

(3.6)

7. \( X_7(a,b_1,b_2;c,d;x,y,z) = \)

\[
\frac{1}{\Gamma(a)} \int_0^\infty e^{-(u)} \phi_0 \left( b_1,b_2;b; y\frac{s}{u}, \frac{zs}{u} \right) \phi_1 \left( -c; x; \frac{s^2}{u^2} \right) ds \]  

(3.7)

8. \( X_8(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = \)

\[
\int_0^\infty e^{-(u)} \left( \frac{s}{u} \right)^{a-1} \phi_1 \left( -c_1; x; \frac{s^2}{u^2} \right) \phi_1 \left( b_1;c_2; y\frac{s}{u} \right) \phi_1 \left( b_2;c_3; \frac{s}{u} \right) ds \]  

(3.8)

9. \( X_9(a,b;c;x,y,z) = \)

\[
\frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(u+v)} \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \sum_{n=0}^{\infty} \frac{1}{c_m!} \left( \frac{x(s^2 + \frac{ys}{u^2} + \frac{t}{uv})^m}{m!} \right) dsdt \]  

(3.9)
10. $X_{10}(a,b;c,d;x,y,z) = \frac{1}{uv} \frac{1}{\Gamma(a) \Gamma(b)} \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u x} + yst \right) F_1 \left( \left. \frac{-d}{z} \right| \left( \frac{t}{v} \right)^2 \right) dsdt \hspace{1cm} (3.10)$

11. $X_{11}(a,b;c,d;x,y,z) = \frac{1}{\Gamma(a) \Gamma(b)} \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u x^2 + z^2} \right) F_1 \left( \left. \frac{-c}{x} x^2 + z^2 \right| \frac{-d}{z} \right) dsdt \hspace{1cm} (3.11)$

12. $X_{12}(a,b,c_2,c_3;x,y,z) = \frac{1}{uv} \frac{1}{\Gamma(a) \Gamma(b)} \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u y^2} \right) F_1 \left( \left. \frac{-c_2}{y} y^2 + \frac{t}{v} \right| \frac{-c_3}{z} \right) dsdt \hspace{1cm} (3.12)$

13. $X_{13}(a,b,c;d;x,y,z) = \frac{1}{\Gamma(a) \Gamma(b) \Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u x^2 + y^2 z^2 + z^2} \right) dsdt \hspace{1cm} (3.13)$

14. $X_{14}(a,b,c,d';x,y,z) = \frac{1}{\Gamma(a) \Gamma(b) \Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u y^2} \right) F_1 \left( \left. \frac{-d'}{z} \right| \frac{-d}{z} \right) dsdt \hspace{1cm} (3.14)$

15. $X_{15}(a,b,c;d';x,y,z) = \frac{1}{uv} \frac{1}{\Gamma(a) \Gamma(b) \Gamma(c)} \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u x^2 + y^2} \right) F_1 \left( \left. \frac{-d'}{z} \right| \frac{-d}{z} \right) dsdt \hspace{1cm} (3.15)$

16. $X_{16}(a,b,c,d';x,y,z) = \frac{1}{uv} \frac{1}{\Gamma(a) \Gamma(b) \Gamma(c)} \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u y^2} \right) F_1 \left( \left. \frac{-d'}{z} \right| \frac{-d}{z} \right) dsdt \hspace{1cm} (3.16)$

17. $X_{17}(a,b,c;d_1,d_2,d_3;x,y,z) = \frac{1}{uv} \frac{1}{\Gamma(a) \Gamma(b) \Gamma(c)} \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u x^2} \right) F_2 \left( \left. b \right| \frac{d_1}{d_2} \right) dsdt \hspace{1cm} (3.17)$

18. $X_{18}(a,b,b',c;d;x,y,z) = \frac{1}{uv} \frac{1}{\Gamma(a) \Gamma(b) \Gamma(c) \Gamma(d)} \int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{s}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \left( \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \right) \left( \frac{s}{u x^2} \right) F_1 \left( \left. \frac{-d}{z} \right| \frac{-d}{z} \right) dsdt \hspace{1cm} (3.18)$
19. \( X_{19} \left( a, b, b', c; d, d'; x, y, z \right) = \)
\[
\frac{1}{\Gamma(a) \Gamma(c)} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\left(\frac{u}{v}\right)} \left(\frac{s}{u}\right)^{a-1} \left(\frac{t}{v}\right)^{c-1} F_{1}(-d; \phi_{2} \left( b, b'; d'; \frac{ys}{u}, \frac{zt}{v} \right) ) ds \, dt \tag{3.19}
\]

20. \( X_{20} \left( a, b, b', c; d, d'; x, y, z \right) = \)
\[
\frac{1}{\Gamma(a) \Gamma(b') \Gamma(c)} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\left(\frac{s}{u} + \frac{t}{v} + \frac{z}{w}\right)} \left(\frac{s}{u}\right)^{a-1} \left(\frac{t}{v}\right)^{b-1} \left(\frac{z}{w}\right)^{c-1} F_{1}(-d; \phi_{2} \left( b, b'; d'; \frac{ys}{u}, \frac{zt}{v} \right) ) ds \, ds \, dt \tag{3.20}
\]

4. Derivation

In order to derive the main results, first we shall derive the result (3.1), i.e.
\[
X_{1} \left( a, b; c, d; x, y, z \right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p} (b)_{p} x^{m} y^{n} z^{p}}{(c)_{m} (d)_{n+p} m! n! p!}
\]

\[
= \frac{1}{\Gamma(a) \Gamma(b)} \sum_{m,n,p=0}^{\infty} \frac{x^{m} y^{n} z^{p}}{(c)_{m} (d)_{n+p} m! n! p!} \int_{0}^{\infty} e^{-\left(\frac{s}{u}\right)} \int_{0}^{\infty} e^{-\left(\frac{t}{v}\right)} \int_{0}^{\infty} e^{-\left(\frac{z}{w}\right)} \left(\frac{s}{u}\right)^{2m+2n+p+a-1} \left(\frac{t}{v}\right)^{p+b-1} \left(\frac{z}{w}\right)^{c-1} ds \, dt \, ds \, dt \tag{4.1}
\]

Proof:

To derive (4.1), we consider L.H.S as I

\[
I = X_{1} \left( a, b; c, d; x, y, z \right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p} (b)_{p} x^{m} y^{n} z^{p}}{(c)_{m} (d)_{n+p} m! n! p!}
\]

\[
= \frac{1}{\Gamma(a) \Gamma(b)} \sum_{m,n,p=0}^{\infty} \frac{x^{m} y^{n} z^{p}}{(c)_{m} (d)_{n+p} m! n! p!} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \left(\frac{s}{u}\right)^{2m+2n+p+a-1} \left(\frac{t}{v}\right)^{p+b-1} \left(\frac{z}{w}\right)^{c-1} ds \, dt \, ds \, dt
\]

Put \( n = n - p \), we get

\[
I = \frac{1}{\Gamma(a) \Gamma(b)} \sum_{m,n,p=0}^{\infty} \frac{x^{m} y^{n-p} z^{p}}{(c)_{m} (d)_{n-p} m! (n-p)! p!} \int_{0}^{\infty} \int_{0}^{\infty} \left(\frac{s}{u}\right)^{2m+2n-2p+a-1} \left(\frac{t}{v}\right)^{b+p-1} \left(\frac{z}{w}\right) \frac{ds}{u} \frac{dt}{v}
\]

\[
= \frac{1}{\Gamma(a) \Gamma(b)} \sum_{m,n,p=0}^{\infty} \frac{x^{m} y^{n-p} z^{p}}{(c)_{m} (d)_{n} m! (n-p)! p!} \int_{0}^{\infty} \int_{0}^{\infty} \left(\frac{s}{u}\right)^{2m+2n-a-1} \left(\frac{t}{v}\right)^{b+p-1} \frac{ds}{u} \frac{dt}{v}
\]
Using the formula \( \frac{1}{(n-p)!} = \frac{(-n)^p}{n!} \) we have

\[
I = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \sum_{m,n=0}^{\infty} \frac{x^m y^n (-p)^p}{m! n! \Gamma(m) \Gamma(n)} \int_0^\infty \int_0^\infty e^{-(\frac{t}{u} + \frac{s}{v})} \left( \frac{z}{u} \right)^{2m + 2n + a - 1} \left( \frac{t}{v} \right)^{b-1} \left( \frac{ds}{u} \right) \left( \frac{dt}{v} \right)
\]

\[
= \frac{1}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \sum_{m,n=0}^{\infty} \frac{x^m y^n (-p)^p}{m! n! \Gamma(m) \Gamma(n)} \int_0^\infty \int_0^\infty e^{-(\frac{t}{u} + \frac{s}{v})} \left( \frac{z}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m! n! \Gamma(m) \Gamma(n)} \left( \frac{z}{u} \right)^{2m + 2n + a - 1} \left( \frac{z}{u} \right)^{b-1} \left( \frac{z}{u} \right)^{2n}
\]

Changing the order of series and integration, we get

\[
I = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(\frac{t}{u} + \frac{s}{v})} \left( \frac{z}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m! n! \Gamma(m) \Gamma(n)} \left( \frac{z}{u} \right)^{2m + 2n + a - 1} \left( \frac{z}{u} \right)^{b-1} \left( \frac{z}{u} \right)^{2n} ds dt
\]

\[
= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(\frac{t}{u} + \frac{s}{v})} \left( \frac{z}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m! n! \Gamma(m) \Gamma(n)} \left( \frac{z}{u} \right)^{2m + 2n + a - 1} \left( \frac{z}{u} \right)^{b-1} \left( \frac{z}{u} \right)^{2n} ds dt
\]

\[
= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(\frac{t}{u} + \frac{s}{v})} \left( \frac{z}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \sum_{m=0}^{\infty} \frac{x^m y^n}{m! \Gamma(m)} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{u} \right)^{2m + 2n + a - 1} \left( \frac{z}{u} \right)^{b-1} \left( \frac{z}{u} \right)^{2n} ds dt
\]

\[
= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(\frac{t}{u} + \frac{s}{v})} \left( \frac{z}{u} \right)^{a-1} \left( \frac{t}{v} \right)^{b-1} \sum_{m=0}^{\infty} \frac{x^m y^n}{m! \Gamma(m)} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{u} \right)^{2m + 2n + a - 1} \left( \frac{z}{u} \right)^{b-1} \left( \frac{z}{u} \right)^{2n} ds dt
\]

\[
= \text{R.H.S}
\]
Special Case:

If we put \( u = 1 \) and \( v = 1 \) in (4.1) then we get

\[
X_1 (a, b; c, d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t} s^{a-1} t^{b-1} \binom{-c}{xs^2} \binom{-d}{ys^2 + zst} dsdt
\]

(4.2)

This verifies the Exton result. Rest could be verified in the same manner.

Acknowledgements

Authors are thankful to the three independent reviewers for the fruitful comments and suggestions which improves the quality of the work.

References


Instruction to Authors
Nepal Journal of Mathematical Sciences (NJMS)

Scope:
Nepal Journal of Mathematical Sciences (NJMS) is the official peer-reviewed journal published by the School of Mathematical Sciences, Tribhuvan University. It is devoted to publish the original research papers as well as critical survey articles related to all branches of pure and applied mathematical sciences. The journal aims to reflect the latest developments in issues related to Applied Mathematics, Mathematical Modelling, Industrial Mathematics, Biomathematics, Pure and Applied Statistics, Applied Probability, Operations Research, Theoretical Computer Science, Information Technology, Data Science, Financial Mathematics, Actuarial Science, Mathematical Economics and many more disciplines. NJMS is published in English and it is open to authors around the world regardless of the nationality. It is published two times in a year. NJMS articles are freely available on online and print form.

Editorial Policy:
NJMS welcomes high-quality original research papers and survey articles in all areas of Mathematical Sciences and real-life applications at all levels including new theories, techniques and applications to science, industry and society. The authors should justify that the theoretical as well as computational results they claim really contribute to Mathematical Sciences or in real-life applications. A complete manuscript be no less than 5 pages and no more than 20 pages (10 pt, including figures, tables, and references) as mentioned in the paper Template. However, this can be extended with the acceptance of the respective referees and the Editor-in-Chief. The submitted manuscript should meet the standard of the NJMS.

Manuscript Preparation:
Manuscript must be prepared in Microsoft Word format (see Template) and LATEX (see template) in the prescribed format with given page limit.

Title of the Paper: The title of the paper should be concise, specific, not exceeding 20 words in length.

Authors Names and Institutional Affiliations: This should include the full names, institutional addresses and email addresses of authors. The corresponding author should also be indicated.

Abstract and Keywords: Each article is to be preceded by an abstract up to maximum 250 words with 4-6 key pertinent words.

Introduction: The introduction briefly describes the problem, purpose, significance and output of the research work, including hypotheses being tested. The current state of the research field should be reviewed carefully and key publications should be cited.

Materials and Methods: It describes the research plan, the materials (or subjects), and the method used. New methods and protocols should be described in detail. It explains in detail the data, sample and population, and the variables used.

Results and Discussions: Results section should provide details of all of the experiments that are required to support the conclusions. Discussion can also be combined with results.

Figures and Tables: Figures and Tables are to be separately numbered, titled in the text serially.

Conclusions and acknowledgments: Not mandatory, but can be added to the manuscript.

References:

Journal Citation:  

Doctoral Dissertation Citation:  

Research Book Citation:  

Manuscript Submission
At the time of submission, authors are requested to include the list of 3-5 panellists of experts of the related research area. All information, submissions and correspondence of the articles should be addressed to

Editor-in-Chief:
Prof. Dr. Narayan Prasad Pahari
Director, School of Mathematical Sciences,
Tribhuvan University, Kathmandu, Nepal

Email: njmseditor@gmail.com, nppahari@gmail.com
CONTENTS

1. A Novel Prey-Predator Quadratic Harvesting Model via Optimal Control Theory and Hopf Bifurcation
   □ Prabir Panja and Dipak Kumar Jana

2. Existence and Extremal Solution of Boundary Value Problem for Nonlinear Hybrid Fractional Differential Equation in Banach Algebras
   □ B.D. Karande and Pravin M. More

3. A New Two - Parameter Lindley Distribution
   □ Rama Shanker and Umme Habibah Rahman

4. 3n+1 Problem and its Dynamics
   □ Bishnu Hari Subedi and Ajaya Singh

5. An Alternative Proof of Rubin's Lemma
   □ Santosh Ghimire

6. Determinants of Households' Adaptation Practices against Climate Change Impact on Off-farm Activities in Western Hill of Nepal
   □ Ananta Raj Dhungana, Vikash Kumar KC, Purna Bahadur Khand and Surya Mani Dhungana

7. On The Degree of Approximation of a Function by Nörlund Means of its Fourier Laguerre Series
   □ Suresh Kumar Sahani, Vishnu Narayan Mishra & Narayan Prasad Pahari

8. Basic Operations on Vedic Mathematics: A Study on Special Parts
   □ Krishna Kanta Parajuli

9. A Note on Natural Transformation of Exton’s Triple series Hypergeometric Function
   □ Harsh Vardhan Harsh, Puneet Krishna Sharma & Shabana Khan

MAILING ADDRESS:
Nepal Journal of Mathematical Sciences
School of Mathematical Sciences
Tribhuvan University, Kirtipur,Kathmandu, Nepal
Email: nppahari@gmail.com, njmseditor@gmail.com