An Alternative Proof of Rubin's Lemma

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Abstract: *Rubin's Lemma is an inhomogenous type inequality which is satisfied by the sequence of dyadic martingales. In this paper, we give a proof using the measure theoretic approach which is simpler and different than the original probabilistic approach.*

Keywords: *Martingales, Law of the iterated logarithm, Quadratic characteristic, Dyadic intervals*

1. Introduction

Rubin's Lemma is an inequality associated to a sequence of dyadic martingales and plays a vital role in the law of the iterated logarithm of dyadic martingales and harmonic functions. These law of the iterated logarithm give the asymptotic behaviour of the sequence of both functions: dyadic martingales and harmonic functions. We can study the role of Rubin's Lemma in the asymptotic behaviour of these functions in the various papers, for instances we refer a few [2] and [5]. In these papers, we find applications of the lemma in order to estimate the size of various sets in the law of the iterated logarithm. There are much more literature available related to the law of the iterated logarithm in various contexts, for instances [1], [3], [4], [6] and [8]. The proof of the Lemma can be found in [2] in which the proof uses the probabilistic approach. In the present work, we give a proof of the Lemma using the measure-theoretic approach. The original proof uses the probabilistic tools of statistics whereas our method uses the simple tools of measure theory which can be easily followed and thus the new proof is simpler in comparison with the original proof. Moreover, our method provides the proof of the lemma in the setting of mathematical analysis.

2. Definitions and Notations

Before proceeding with the main work, we first fix some notations and give some basic definitions which will be used in the course of proof.

A dyadic subinterval of the unit interval $[0, 1]$ is an interval of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ where $n = 0,1,2,\ldots$ and $j = 0,1,\ldots,2^n - 1$.

A dyadic martingale is a sequence of integrable functions $\{f_n\}_{n=0}^{\infty}$ where $f_n: [0,1) \to \mathbb{R}$ such that

    a. for every n, f_n is $\mathcal{F}_n$ measurable where $\mathcal{F}_n$ is the sigma algebra generated by the dyadic intervals of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ for $j = 0,1,\ldots,2^n - 1$.

    b. conditional expectation $E(f_{n+1}|\mathcal{F}_n) = f_n$ where
$$E(f_{n+1}|\mathcal{F}_n)(x) = \frac{1}{|Q|} \int_{Q_n} f_n(y)dy \text{ with } |Q_n| = \frac{1}{2^n}, x \in Q_n.$$

Associated with a sequence of dyadic martingales $\{f_n\}_{n=0}^{\infty}$, we define:

i.   increments: $d_k = f_k - f_{k+1}$. So $f_n(x) = \sum_{k=1}^{n} d_k(x) + f_0$.

ii.   quadratic characteristics or square function: $S_n^2 f(x) = \sum_{k=1}^{n} d_k^2(x)$.

iii.   limit function $S^2 f(x) = \lim_{n \to \infty} S_n^2 f(x) = \sum_{k=1}^{\infty} d_k^2(x)$.

We now state the inequality associated to the dyadic martingales and this inequality is called Rubin's Lemma according to **[7]**.

Rubin's Lemma: For a dyadic martingales $\{f_n\}_{n=0}^{\infty}$ with $f_0 = 0$, we have

$$\int_0^1 \exp\left(f_n(x) - \frac{1}{2} S_n^2 f(x) \right) dx \leq 1.$$

We now give the proof the lemma using a different approach than the original proof.

3. Proof of the Main Result

Let us first define

$$g(n) = \int_0^1 \exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) dx.$$

We now claim that the function $g(n)$ is a decreasing function of n. Let Q_{nj} be an arbitrary n^{th} generation dyadic interval. One can see that $\sum_{k=0}^{n} d_k(x) = f_n$ and f_n is constant on Q_{nj}.
Using this fact, we have

$$g(n+1) = \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp\left(\sum_{k=0}^{n+1} d_k(x) - \frac{1}{2} \sum_{k=0}^{n+1} d_k^2(x) \right) dx.$$

$$= \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \exp\left(d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx.$$

$$= \sum_{j=0}^{2^n} \left[\exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \right]_{Q_{nj}} \int_{Q_{nj}} \exp\left(d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right)$$

Let $Q'_{(n+1)j}$ and $Q''_{(n+1)j}$ be the dyadic subintervals of Q_{nj}. Suppose d_{n+1} takes the value α on $Q'_{(n+1)j}$.
Then by expectation condition, d_{n+1} takes the value $-\alpha$ on $Q''_{(n+1)j}$.

This gives

$$\int_{Q_{nj}} \exp\left(d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right)$$

$$= \int_{Q'_{(n+1)j}} \exp\left(d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx + \int_{Q''_{(n+1)j}} \exp\left(d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx$$

$$= \int_{Q'_{(n+1)j}} \exp\left(\alpha - \frac{1}{2}\alpha^2\right) dx + \int_{Q''_{(n+1)j}} \exp\left(-\alpha - \frac{1}{2}\alpha^2\right) dx$$

$$= \left[\exp\left(\alpha - \frac{1}{2}\alpha^2\right) + \exp\left(-\alpha - \frac{1}{2}\alpha^2\right)\right] \frac{1}{2^{n+1}}$$

$$= 2\exp\left(-\frac{\alpha^2}{2}\right)\left(\frac{e^\alpha + e^{-\alpha}}{2}\right) \frac{1}{2^{n+1}}$$

Thus, $\int_{Q_{nj}} \exp\left(d_{n+1}(x) - \frac{1}{2}d_{n+1}^2(x)\right) = 2\exp\left(-\frac{\alpha^2}{2}\right)\cosh\alpha \frac{1}{2^{n+1}}$.

Now using the elementary fact that $\cosh x \le \exp\left(\frac{x^2}{2}\right)$,we have

$$g(n+1) \le \sum_{j=0}^{2^n}\left[\exp\left(\sum_{k=0}^n d_k(x) - \frac{1}{2}\sum_{k=0}^n d_k^2(x)\right)\right]_{Q_{nj}} 2\exp\left(-\frac{\alpha^2}{2}\right)\exp\left(\frac{\alpha^2}{2}\right)\frac{1}{2^{n+1}}$$

$$g(n+1) \le \sum_{j=0}^{2^n}\left[\exp\left(\sum_{k=0}^n d_k(x) - \frac{1}{2}\sum_{k=0}^n d_k^2(x)\right)\right]_{Q_{nj}} |Q_{nj}|$$

$$g(n+1) \le \sum_{j=0}^{2^n}\int_{Q_{nj}} \exp\left(\sum_{k=0}^n d_k(x) - \frac{1}{2}\sum_{k=0}^n d_k^2(x)\right) dx$$

$$g(n+1) \le g(n).$$

Let Q_{11} and Q_{12} be the dyadic subintervals of Q_0. Assume that d_1 takes value θ on Q_{11} so that it takes value $-\theta$ on Q_{12}.

Now $g(1) = \int_0^1 \exp\left(d_1(x) - \frac{1}{2}d_1^2(x)\right) dx$

$$= \int_0^{\frac{1}{2}} \exp\left(\theta - \frac{1}{2}\theta^2\right) dx + \int_{\frac{1}{2}}^1 \exp\left(-\theta - \frac{1}{2}\theta^2\right) dx$$

$$= \exp\left(\theta - \frac{1}{2}\theta^2\right)\frac{1}{2} + \exp\left(-\theta - \frac{1}{2}\theta^2\right)\frac{1}{2}$$

$$= \exp\left(-\frac{1}{2}\theta^2\right)\frac{(e^\theta + e^{-\theta})}{2}$$

$$= \exp\left(-\frac{1}{2}\theta^2\right)\cosh\theta$$

$$g(1) \le \exp\left(-\frac{1}{2}\theta^2\right)\exp\left(\frac{1}{2}\theta^2\right) = 1.$$

Since $g(n)$ is decreasing and $g(1) \le 1$, we conclude that $g(n) \le 1$.
This gives

$$\int_0^1 \exp\left(\sum_{k=0}^n d_k(x) - \frac{1}{2}\sum_{k=0}^n d_k^2(x)\right) dx \le 1.$$

Hence, $\int_0^1 \exp\left(f_n(x) - \frac{1}{2} S_n^2 f(x)\right) dx \leq 1.$

This proves our result.

Finally we note that if we rescale the function by some λ, we have

$$\int_0^1 \exp\left(\lambda f_n(x) - \frac{1}{2} \lambda^2 S_n^2 f(x)\right) dx \leq 1.$$

This shows that the Rubin's Lemma is an inhomogeneous type inequality.

Conclusion

In this paper, we derived an inequality associated to a sequence of dyadic martingales which is popularly known as Rubin's Lemma. We used the measure theoretic approach to prove the inequality which is different than the original probabilistic approach.

Acknowledgement

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