

On the Degree of Approximation of a Function by Nörlund Means of its Fourier Laguerre Series

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Abstract: In this paper, we have proved the degree of approximation of function belonging to $L[0, \infty)$ by Nörlund Summability of Fourier-Laguerre series at the end point $x = 0$. The purpose of this paper is to concentrate on the approximation relations of the function in $L[0, \infty)$ by Nörlund Summability of Fourier-Laguerre series associate with the given function motivated by the works [3], [9] and [13].

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1. Introduction

The concepts of product summability methods are more powerful than the individual summability methods and it gives an approximation for wider class of functions than the individual methods [6]. A bulk number of researchers have studied the degree of approximation of a function using different summability means of its Fourier-Laguerre series [1], [5], [7], [10], [11], [14] and [15]. This can be done at the point after replacing the continuity condition in Szegö theorem by much lighter conditions.

Let $f(t)$ be a Lebesgue measurable function in the interval $(0, \infty)$ such that the integral

$$\int_0^{\infty} e^{-x} x^{\alpha} f(x) L_n^{(\alpha)}(x) dx, \quad \alpha > -1 \quad (1.1)$$

exists, where $L_n^{(\alpha)}(x)$ is the n^{th} Laguerre polynomial of order $\alpha > -1$ defined the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \omega^n = (1 - \omega)^{-\alpha-1} e^{\frac{-x\omega}{1-\omega}} \quad (1.2)$$

The Fourier series of the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \quad \alpha > -1, \quad (1.3)$$

where the coefficients a_n are the defined by the following formulae

$$\Gamma(\alpha + 1) A_n^{\alpha} a_n = \int_0^{\infty} e^{-x} x^{\alpha} f(x) L_n^{(\alpha)}(x) dx \quad (1.4)$$

where $A_n^\alpha = \binom{n+a}{n} \sim n^\alpha$.

Also, we write
$$\phi(x) = \frac{1}{\Gamma(\alpha+1)} e^{-x} x^\alpha \tag{1.5}$$

Let $\sum u_n$ be a given infinite series with the sequence $\{S_n\}$ of its partial sums. Let $\{p_n\}$ be a sequence of real or complex constants with p_n as its non-vanishing n^{th} partial sum. The sequence to sequence transformation is given by

$$\tau_n = \frac{1}{p_n} \sum_{m=0}^n p_{n-m} S_m. \tag{1.6}$$

which defines the sequence of Nörlund means of the sequence $\{S_n\}$, generated by the sequence of coefficients $\{p_n\}$. If $\tau_n \rightarrow s$ then $\sum u_n$ or the sequence $\{S_n\}$ to the sum S . If the sequence of Nörlund mean $\{\tau_n\}$ is of bounded variation i.e. if

$$|\tau_n - \tau_{n-1}| < \infty \tag{1.7}$$

then the series $\sum u_n$ is said to be absolute Nörlund summable [5], [6] and [12]. The absolute Nörlund summability of a Fourier series has been studied by several researchers like Bor [5], [7], Mazhar [10], Fadden [11], Padhy, Tripathi and Mishra [14] and Siddiqui, Gi, Bohar and Brono [15].

In 1976, Yadav [19] was the first to establish a result on absolute Nörlund summability of Laguerre series at origin and in 1979, Beohar and Jadiya [2] established a result on the absolute Nörlund summability of Laguerre series at the point $x = 0$. Later on, several researchers like Alghamdi & Mursaleen [1], Beohar and Jadiya [2], [3], Khatri and Mishra [8], Karasniqi [9], Nigam and Sharma [13], Shanker [16], Tiwari and Kachhara [18] obtained the degree of approximations of $L[0, \infty)$ of the Fourier-Laguerre series by Cesaro mean Nörlund, Euler, $(C,1)(E,q), (C,2)(E,q)$ harmonic-Euler mean et al. Concerning the absolute Nörlund summability of Laguerre series Yadav [19] has proved the following theorem:

Theorem 1.1: If $\{p_n\}$ be a non-negative and non-increasing sequence of constant such that

$P_n = \sum_{r=0}^n p_r \rightarrow \infty$ and $\sum_n \frac{n^{\frac{(-1+2\alpha)}{4}}}{p_n}$ is convergent. i.e. $\sum_n \frac{n^{\frac{(-1+2\alpha)}{4}}}{p_n} < \infty$, then the Fourier series (3) is $|N, p_n|$ summable at the end point $x = 0$, provided that for some small positive ϵ ,

$$\phi(x) \in BV[0, \infty) \tag{1.8}$$

where

$$\int_n^\infty e^{-\frac{x}{2}} x^{\frac{\alpha}{2} - \frac{7}{12}} |f(x)| dx = o\left(n^{-\frac{1}{2}}\right) \text{ and } F(x) \text{ is bounded in } [0, \epsilon]. \tag{1.9}$$

Where,

$$\phi(x) = [f(x) - f(0)] \cdot \frac{e^{-x} x^\alpha}{\Gamma(\alpha+1)} \tag{1.10}$$

and,

$$F(x) = \frac{[f(x) - f(0)] \cdot e^{-x} x^{\frac{(-1+2\alpha)}{4}}}{\Gamma(\alpha+1)} \tag{1.11}$$

2. Main Results

Before proceeding with the main result, we begin with recalling some Lemmas that are required to prove main theorem in this paper.

Lemma 2.1: (Szegő [17], p.177) Let α be arbitrary and real, c and w be fixed positive constants, and let $n \rightarrow \infty$. Then

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\frac{\alpha-1}{2}} \cdot o\left(n^{\frac{2\alpha-1}{4}}\right), & \text{if } \frac{c}{n} \leq x \leq w \\ o(n^\alpha), & \text{if } 0 \leq x \leq \frac{c}{n} \end{cases} \quad (2.1)$$

Lemma 2.2: (Szegő[17], p. 240) Let α be arbitrary and real, $w > 0$, $0 < \eta < 4$. Then we have for $n \rightarrow \infty$,

$$\max. e^{\frac{-x}{2}} \cdot x^{\frac{2\alpha+1}{4}} \cdot |L_n^{(\alpha)}(x)| \sim \begin{cases} n^{\frac{\alpha-1}{2}}, & \text{if } w \leq x \leq (4-\eta) \cdot n \\ n^{\frac{\alpha-1}{2}}, & \text{if } x \geq w \end{cases} \quad (2.2)$$

Lemma 2.3: (Bhatta [4]) Let $\sum a_n$ be an infinite series with S_n as its n^{th} partial sum and $\{p_n\}$ be a non-increasing sequence such that $p_n \rightarrow \infty$. If $\sum_n \frac{|S_n|}{p_n} < \infty$ i.e. convergent, then the series $\sum_n a_n$ is $|N, P_n|$ summable.

Considering the above facts, we prove the following theorem:

Theorem 2.4: Let $\chi(x)$ be a non-negative, and non-increasing function of x such that $x^\alpha \cdot \chi\left(\frac{1}{x}\right) \rightarrow 0$ as $n \rightarrow 0$. Let $\{p_n\}$ be a non-negative and monotonic non-increasing sequence of constants with P_n as its non-vanishing n^{th} partial sum such that $\sum_n \frac{|\chi(n)|}{P_n} < \infty$ i.e. convergent. Let $\frac{-1}{2} > \alpha > -1$ and w is a fixed positive constant. If

$$\int_x^w \frac{|\phi(s)|}{S^{\alpha+1}} ds = o\left[\chi\left(\frac{1}{x}\right)\right], \text{ as } x \rightarrow 0, \quad (2.4)$$

$$\int_w^n e^{\frac{x}{2}} \cdot x^{-\frac{\alpha}{2} - \frac{3}{4}} |\phi(x)| dx = o\left[n^{-\frac{\alpha}{2} - \frac{1}{4}} \cdot \chi(n)\right] \quad (2.5)$$

and

$$\int_n^\infty e^{\frac{x}{2}} \cdot x^{-\frac{(6\alpha+7)}{12}} |\phi(x)| dx = o\left[n^{-\frac{(2\alpha+1)}{4}} \cdot \chi(n)\right] \text{ as } n \rightarrow \infty, \quad (2.6)$$

then the Fourier-Laguerre series (1.3) is $|N, p_n|$ summable at the point $x = 0$.

Proof:

First of all , under the hypothesis (2.4), we prove

$$\int_0^x |\phi(s)| dx = o\left[x^{\alpha+1} \cdot \chi\left(\frac{1}{x}\right)\right] \quad (2.7)$$

For this, we have $I(s) = \int_x^w \frac{|\phi(s)|}{S^{\alpha+1}} ds = o\left[\chi\left(\frac{1}{x}\right)\right]$

Then $|\phi(s)| = -I'(s) \cdot S^{\alpha+1}$

Therefore, $\int_0^x |\phi(s)| ds = -\int_0^x S^{\alpha+1} \cdot I'(s) ds$
 $= -[S^{\alpha+1} I(S)]_0^x + (\alpha + 1) \int_0^x S^\alpha \cdot I(s) ds$

$$\begin{aligned}
 &= o \left[S^{\alpha+1} \cdot \chi \left(\frac{1}{s} \right) \right] + o \left[S^{\alpha+1} \cdot \chi \left(\frac{1}{s} \right) \right] \\
 &= o \left[S^{\alpha+1} \cdot \chi \left(\frac{1}{s} \right) \right]
 \end{aligned}$$

Then the proof of the theorem is as follows:

$$\text{Since } L_n^{(\alpha)}(0) = \binom{n+\alpha}{\alpha} \tag{2.8}$$

$$\begin{aligned}
 \text{therefore } S_n(0) &= \sum_{k=0}^n a_k L_k^{(\alpha)}(0) \\
 &= \sum_{k=0}^n \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-x} \cdot x^\alpha f(x) \sum_{k=0}^n L_k^{(\alpha)}(x) dx \\
 &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-x} \cdot x^\alpha f(x) L_n^{(\alpha+1)}(x) dx
 \end{aligned}$$

Again, due to the orthogonality of Laguerre polynomials, we have

$$\begin{aligned}
 S_n(0) - f(0) &= \int_0^\infty \phi(x) L_n^{(\alpha+1)}(x) dx \\
 &= \left(\int_0^{\frac{c}{n}} + \int_{\frac{c}{n}}^w + \int_w^n + \int_n^\delta \right) \phi(x) L_n^{(\alpha+1)}(x) dx \\
 &= I_1 + I_2 + I_3 + I_4 \text{ (say)}
 \end{aligned} \tag{2.9}$$

First we consider I_1 ,

$$\begin{aligned}
 |I_1| &= \int_0^{\frac{c}{n}} |\phi(x)| L_n^{\alpha+1}(x) dx \\
 &= o(n^{\alpha+1}) \int_0^{\frac{c}{n}} |\phi(x)| dx \\
 &= o[\chi(n)], \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{2.10}$$

Next we consider I_2 , using (2.1) and (2.4), we have

$$\begin{aligned}
 |I_2| &= o \left(n^{\frac{\alpha+1}{2}} \right) \int_{\frac{c}{n}}^w |\phi(x)| x^{\frac{\alpha+3}{4}} dx \\
 &= o \left(n^{\frac{\alpha+1}{2}} \right) \int_{\frac{c}{n}}^w \frac{|\phi(x)|}{x^{\alpha+1}} \left(x^{\frac{\alpha+1}{2}} \right) dx \\
 &= \left(n^{\frac{\alpha+1}{2}} \right) o \left(n^{-\frac{\alpha}{2} - \frac{1}{4}} \right) \int_{\frac{c}{n}}^w \frac{|\phi(x)|}{x^{\alpha+1}} dx \\
 &= o(1) o[\chi(n)] \\
 &= 0[\chi(n)], \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{2.11}$$

Considering I_3 and using (2.2), (2.3) and (2.5), we have

$$\begin{aligned}
 |I_3| &= \int_w^n |\phi(x)| \left| L_n^{(\alpha+1)}(x) \right| dx \\
 &= \int_w^n |\phi(x)| e^{\frac{x}{2}} x^{\frac{-(2\alpha+3)}{4}} \phi \left(n^{\frac{2\alpha+1}{4}} \right) dx \\
 &= o \left(n^{\frac{2\alpha+1}{4}} \right) \int_w^n e^{\frac{x}{2}} x^{\frac{-(2\alpha+1)}{4}} |\phi(x)| dx
 \end{aligned}$$

$$\begin{aligned}
 &= o\left(n^{\frac{2\alpha+1}{4}}\right) o\left(n^{\frac{-(2\alpha+1)}{4}} \chi(n)\right) \\
 &= o[\chi(n)], \text{ as } n \rightarrow \infty
 \end{aligned} \tag{2.12}$$

Again, considering I_4 , using (2.2), (2.6) and (2.7), we have

$$\begin{aligned}
 |I_4| &= \int_n^\infty |\phi(x)| e^{\frac{x}{2}} x^{\frac{-(2\alpha+3)}{4}} o\left(n^{\frac{6\alpha+5}{12}}\right) dx \\
 &= o\left(n^{\frac{6\alpha+5}{12}}\right) \int_n^\infty e^{\frac{x}{2}} x^{\frac{-(2\alpha+3)}{4}} |\phi(x)| dx \\
 &= o\left(n^{\frac{6\alpha+5}{12}}\right) \int_n^\infty \left(e^{\frac{x}{2}} x^{\frac{-(6\alpha+7)}{12}} x^{\frac{-1}{6}} |\phi(x)| dx\right) \\
 &= o\left(n^{\frac{6\alpha+5}{12}}\right) o\left(n^{\frac{-1}{6}}\right) \int_n^\infty e^{\frac{x}{2}} x^{\frac{-(6\alpha+7)}{12}} |\phi(x)| dx \\
 &= o\left(n^{\frac{2\alpha+1}{4}}\right) o\left(n^{\frac{-(2\alpha+1)}{4}} \chi(n)\right) \\
 &= o[\chi(n)], \text{ as } n \rightarrow \infty
 \end{aligned} \tag{2.13}$$

Now combining (2.10), (2.11), (2.12) and (2.13), and using $f(0) = 0$, we get

$$|S_n(0) - 0| = o[\chi(n)], \text{ as } n \rightarrow \infty \text{ and therefore } |S_n| = o[\chi(n)], \text{ as } n \rightarrow \infty.$$

Therefore applying (2.7), we have

$$\sum_n \frac{|S_n|}{P_n} = \sum_n \frac{|\chi(n)|}{P_n} < \infty \text{ i.e. convergent.}$$

This completes the proof of the theorem.

Conclusion

In this paper, we have proved a theorem related to the degree of approximation of function belonging to $L[0, \infty)$ by Nörlund Summability of Fourier-Laguerre series at the end point $x = 0$. This work establishes some of the results that characterize the approximation relations of the function in $L[0, \infty)$ by Nörlund Summability of Fourier-Laguerre series. In fact, these results can be used for further study in many practical problems in science and engineering.

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